(Section 1) Curves

(subsection 1.2.) Parametrized Curves

(Def) R^{3} is denoted the set of triples (x,y,z) of real numbers. A real function of a real variable is differentiable (or smooth) if it has, at all points, derivatives of all orders.

(Def) A parametrized differentiable curve is a differentiable map \alpha : I \to R^{3} of an open interval I = (a,b) of the real line R into R^{3}. The differentiable means that \alpha is a correspondence which maps each t \in I into a point \alpha(t) = (x(t), y(t), z(t)) \in R^{3} which the function x(t), y(t), z(t) are differentiable.

(Def) the vector (x’(t), y’(t), z’(t)) = \alpha ‘(t) \in R^{3} is called the tangent vector (or velocity vector) of the curve \alpha at t. The image set \alpha (I) \subset R^{3} is called the trace of \alpha.

(Def) Let u = (u\_{1}, u\_{2}, u\_{3}) \in R^{3} and define its norms (or length) by |u| = \sqrt{u\_{1}^{2} + u\_{2}^{2} + u\_{3}^{2}}. Let v = (v\_{1}, v\_{2}, v\_{3}) \in R^{3}, and let \theta , 0 \le \theta \le \pi be the angle formed by the segments 0u and 0v. the inner product u \cdot v is defined by u \cdot v = |u||v| cos \theta.

(prop)

Assume that u and v are nonzero vectors. Then u \cdot v = 0 iff u is orthogonal to v.

u \cdot v = v \cdot u.

\lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v.

u \cdot (v + w) = u \cdot v + u \cdot w.

(Def) Let e\_{1} = (1,0,0) , e\_{2} = (0,1,0) , and e\_{3} = (0,0,1) . e\_{i} \cdot e\_{j} = 1 if i = j and that e\_{i} \cdot e\_{j} = 1 if i = j and that e\_{i} \cdot e\_{j} = 0 if i \neq j , where i , j = 1,2,3. Thus, by writing u = u\_{1}e\_{1} + u\_{2}e\_{2} + u\_{3}e\_{3} , v = v\_{1}e\_{1} + v\_{2}e\_{2} + v\_{3}e\_{3} , we obtain u \cdot v = u\_{1}v\_{1} + u\_{2}v\_{2} + u\_{3}v\_{3}.

(subsection 1.3.) Regular Curves ; Arc length

(Def) Let \alpha : I \to R^{3} be a parametrized differentiable curve. For each t \in I where \alpha ‘(t) \neq 0, there is a well-defined straight line which contains the point \alpha (t) and the vector \alpha ‘(t) . This line is called the tangent line to \alpha at t. We call any point where \alpha’(t) = 0 a singular point. We restrict our attention to curves without singular points.

(Def) A parametrized differentiable curve \alpha : I \to R^{3} is said to be regular if \alpha ‘(t) \neq 0 for all t \in I.

Given t \in I, the arc length of a regular parametrized curve \alpha : I \to R^{3} , from the point t\_{0}, is by definition s(t) = \int\_{t\_{0}}^{t} |\alpha ‘(t)| dt, where |\alpha ‘(t)| = \sqrt{(x’(t))^{2} + (y’(t))^{2} + (z’(t))^{2}} is the length of the vector \alpha ‘(t).

(Def) Given the curve \alpha parametrized by arc length s \in (a,b) , we may consider the curve \beta defined in (-b, -a) by \beta(-s) = \alpha (s) , which has the same trace as the first one but is described in the opposite direction. We say that these two curves differ by a change of orientation.

(subsection 1.4.) The Vector product in R^{3}

(Def) Two ordered basis e = {e\_{i}} and f = {f\_{i}} , i = 1,…,n, of an n-dimensional vector space V has the same orientation if the matrix of change of basis has positive determinant. We denote this relation by e ~ f.

e~f is an equivalence relation. Each of equivalence classes determined by the above relation is called an orientation of V. In the case V = R^{3}, there exists a natural ordered basis e\_{1} = (1,0,0) , e\_{2} = (0,1,0) , and e\_{3} = (0,0,1). We call the orientation corresponding the positive orientation of R^{3}, the other one being the negative orientation.

(Def) Let u, v \in R^{3} . The vector product of u and b is the unique vector u \wedge v \in R^{3} characterized by (u \wedge v) \cdot w = det (u,v,w) for all w \in R^{3}. Here det (u,v,w) means that if we express u,v, and w in the natural basis {e\_{i}} , u = \sum u\_{i}e\_{i} , v = \sum v\_{i}e\_{i} , w = \sum w\_{i}e\_{i} , i = 1,2,3, then det(u,v) = \begin{vmatrix} u\_{1}& u\_{2} &u\_{3} \\ v\_{1}& v\_{2} &v\_{3} \\ w\_{1} &w\_{2} &w\_{3} \end{vmatrix}. Usually u \wedge v is written u \times v and denoted as cross product.

(prop) u \wedge v = - v \wedge u.

u \wedge v depends linearly on u and v; for any real numbers a,b, we have (au + bw) \wedge v = au \wedge v + bw \wedge v.

u \wedge v = 0 iff u and v are linearly dependent.

(u \wedge v) \cdot u = 0, (u \wedge v) \cdot v = 0.

The vector product of u and v is a vector u \wedge v perpendicular to a plane generated by u and v, with a norm equal to the area of a parallelogram generated by u and v and a direction s.t. (u,v, \u \wedge v) is a positive basis.

(subsection 1.5.) The Local Theory of Curves Parametrized by Arc Length

(Def) Let \alpha : I \to R^{3} be a curve parametrized by arc length s \in I. The number |\alpha ‘’(s) | = k(s) is called the curvature of \alpha at s.

(Def) At points where k(s) \neq 0, a unit vector n(s) in the direction \alpha’’(s) is well defined by the equation \alpha’’(s) = k(s) n(s). n(s) is normal to \alpha’(s) and is called the normal vector at s. The plane determined by the unit tangent and normal vectors, \alpha’(s) and n(s), is called the osculating plane at s.

s \in I is a singular point of order 1 if \alpha’’(s) = 0. We denote t(s) = \alpha’(s) the unit tangent vector of \alpha at s. Thus, t’(s) = k(s)n(s).

The unit vector b(s) = t(s) \wedge n(s) is normal to the osculating plane and will be called the binormal vector at s. It measures how rapidly the curve pulls away from the osculating plane at s , in a neighborhood of s. b’(s) is parallel to n(s), and we may write b’(s) = \tau(s) n(s) for some function \tau (s).

(Def) Let \alpha : I \to R^{3} be a curve parametrized by arc length s s.t. \alpha’’(s) \neq 0 , s \in I. The number \tau(s) defined by b’(s) = \tau(s) n(s) is called the torsion of \alpha at s.

(Def) To each value of the parameter s, we have associated three orthogonal unit vectors t(s), n(s), b(s) . The trihedrom formed is referred to as the Frenet trihedron at s. We call the equations t’ = kn , n’ = -kt - \tau b, b’ = \tau n the Frenet formulas. The tb plane is called the rectifying plane, and the nb plane the normal plane. The lines which contain n(s) and b(s) and pass through \alpha(s) are called the principal normal and the binormal, respectively. The inverse R = 1/k of the curvature is called the radius of curvature at s.

(Fundamental Theorem of the Local Theory of Curves) Given differentiable functions k(s) >0 and \tau(s), s \in I, there exists a regular parametrized curve \alpha : I \to R^{3} s.t. s is the arc length, k(s) is the curvature, and \tau(s) is the torsion of \alpha. Moreover, any other curve \bar{\alpha} , satisfying the same conditions, differs from \alpha by a rigid motion ; that is , there exists an orthogonal linear map \rho of R^{3}, with positive determinant, and a vector c such that \bar{\alpha} = \rho \bullet \alpha + c.

(subsection 1.6.) The Local canonical form

(Def) Let \alpha : I \to R^{3} be a curve parametrized by arc length without singular points of order 1. We write the equations of the curve, in a neighborhood of s\_{0}, using the trihedron t(s\_{0}) , n(s\_{0}), b(s\_{0}) as a basis for R^{3}. WLOG , s\_{0} = 0. Consider the taylor expansion \alpha(s) = \alpha(0) + s(\alpha’(0)) + \frac{s^{2}}{2} \alpha’’(0) + \frac{s^{3}}{6} \alpha’’’(0) + R where \lim\_{s \to 0} R/s^{3} = 0. Let us now take the system Oxyz in such a way that the origin O agrees with \alpha(0) and that t = (1,0,0) , n = (0,1,0) , b = (0,0,1). Under these conditions, \alpha(s) = (x(s), y(s), z(s)) is given by x(s) = s - \frac{k^{2}s^{2}}{6} + R\_{x} , y(s) = \frac{k}{2} s^{2} + \frac{k’s^{3}}{6} + R\_{y} , z(s) = -\frac{k \tau}{6} s^{3} + R\_{z} ,where R = (R\_{x} , R\_{y}, R\_{z}) . This representation is called the local canonical form of \alpha, in a neighborhood of s = 0.

(prop) Existence of a neighborhood J \subset I of s = 0 s.t. \alpha(J) is entirely contained in the one side of the rectifying plane toward which the vector n is pointing.

The osculating plane at s is the limit position of the plane determined by the tangent line at s and the point \alpha(s + h) when h \to 0.

(subsection 1.7.) Global Properties of Plane Curves

(Def) A differentiable function on a closed interval [a,b] is the restriction of a differentiable function defined on an open interval containing [a,b].

A closed plane curve is a regular parametrized curve \alpha : [a,b] \to R^{2} s.t. \alpha and all its derivatives agree at a and b; that is, \alpha (a) = \alpha (b), \alpha’(a) = \alpha’(b) , …

The curve \alpha is simple if it has no further self-intersections.

We usually consider the curve \alpha : [0, l] \to R^{2} parametrized by arc length s; hence, l is the length of \alpha.

We assume that a simple closed curve C in the plane bounds a region of this plane that is called the interior of C. We assume that the parameter of a simple closed curve can be so chosen that if one is going along the curve in the direction of increasing parameters, the interior of the curve remains left. Such a curve will be called positively oriented.

(subsubsection A) The Isoperimetric Inequality.

(Def) The area A bounded by a positively oriented simple closed curve \alpha (t) = (x(t), y(t)), where t \in [a,b] is an arbitrary parameter : A = -\int\_{a}^{b} y(t) x’(t) dt = \int\_{a}^{b} x(t) y’(t) dt = \frac{1}{2} \int\_{a}^{b} (xy’ – yx’) dt.

(The Isoperimetric Inequality) (Thm 1) Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then l^{2} – 4 \pi A \ge 0 , and equality holds iff C is a circle.

This applies to C^{1} curves, that is, curves \alpha(t) = (x(t), y(t)) , t \in [a,b] for which we require only that the functions x(t), y(t) have continuous first derivatives. It holds for piecewise C^{1} curves, which is continuous curves that are made up by a finite number of C^{1} arcs.

(subsubsection B) The Four-Vertex Theorem  
(Def) Let \alpha : [0,l] \to R^{2} be a plane closed curve given by \alpha(s) = (x(s), y(s)) . Since s is the arc length, the tangent vector t(s) = (x’(s), y’(s)) has unit length. The tangent indicatrix t : [0,l] \to R^{2} that is given by t(s) = (x’(s), y’(s)) is a differentiable curve, the trace of which is contained in a circle of radius 1.

Define a global differentiable function \theta : [0,l] \to R by \theta(s) = \int\_{0}^{s} k(s) ds . Since \alpha is closed, this angle is an integer multiple I of 2 \pi ; that is , \int\_{0}^{l} k(s) ds = \theta(l) - \theta (0) = 2 \pi I. This integer I is called the rotation index of the curve \alpha.

(prop)(Theorem of Turning Tangents) The rotation index of a simple closed curve is \mp 1, where the sign depends on the orientation of the curve.

(Def) A regular, plane curve \alpha : [a,b] \to R^{2} is convex if, for all t \in [a,b] , the trace \alpha([a,b]) of \alpha lies entirely on one side of the closed half-plane determined by the tangent line at t.

A vertex of a regular plane curve \alpha : [a,b] \to R^{2} is a point t \in [a,b] where k’(t) = 0.

(The Four-vertex Theorem) (Thm 2) A simple closed convex curve has at least four vertices.

(Lemma) Let \alpha : [0,1] \to R^{2} be a plane closed curve parametrized by arc length and let A,B, and C be arbitrary real numbers. Then \int\_{0}^{1} (Ax + By + C) \frac{dk}{ds} ds = 0 where the functions x = x(s), y = y(s) are given by \alpha(s) = (x(s), y(s)) , and k is the curvature of \alpha.

(subsubsection C) The Cauchy-Crofton Formula

(Def) Let C be a regular curve in the plane. We look at all straight lines in the plane that meet C and assign to each such line a multiplicity which is the number of its intersection points with C.

A straight line in the plane can be thought as a point in a plane given by two parameters \rho and \theta which determines a line. Define a measure (area) of a subset of straight lines in the plane, by the area of certain plane.

(The Cauchy-Crofton Formula) (Thm 3) Let C be a regular plane curve with length l. The measure of the set of straight lines (counted with multiplicities) which meet C is equal to 2l.

(Def) A rigid motion in R^{2} is a map F : R^{2} \to R^{2} given by (\bar{x}, \bar{y}) \to (x,y) where x = a + \bar{x} cos \phi - \bar{y} sin \phi , y = b + \bar{x} sin \phi + \bar{y} cos \phi.

(Prop 1) Let f (x,y) be a continuous function defined in R^{2}. For any set S \subset R^{2}, define the area A of S by A(S) = \int \int\_{S} f(x,y) dx dy (on only those sets for which the abouve integral exists). Assume that A is invariant under rigid motions; that is , if S is any set and \bar{S} = F^{-1}(S), where F is the rigid motion, we have A(\bar{S}) = \int \int\_{S} f(\bar{x} , \bar{y}) d \bar{x} d \bar{y} = \int \int\_{S} f(x,y) dx dy = A(S). Then f(x,y) = const.

Jacobian of a rigid motion is 1, and the rigid motions are transitive on points of the plane; that is, given two points in the plane there exists a rigid motion taking one point into the other.

(Def) In the set of all straight lines in the plane \mathcal{L} = {(p, \theta) \in R^{2} ; (p,\theta) ~ (p , \theta + 2k\pi) and (p, \theta) ~ (-p, \theta \mp \pi)}, from the prop 1, define the measure of a set \mathcal{S} \subset \mathcal{L} as \int \int\_{\mathcal{S}} d \rho d \theta in the same way as prop 1.

(App) If a curve is not rectifiable but the \int \int n d \rho d \theta (Let n = n(\rho, \theta) be the number of intersection points of the straight line (\rho , \theta)) has a meaning, this can be used to determine the length of such a curve.

Consider a family of parallel straight lines s.t. two consecutive lines are at a distance r. Rotate this family by angles of \pi/4, 2 \pi/4. 3 \pi/4 in order to obtain four families of straight lines. Let n be the number of intersection points of a curve with all these lines. Then \frac{1}{2} nr \frac{\pi}{4} is an approximation to length of C.

(Section 2) Regular Surfaces

(subsection 2.2) Regular Surfaces : Inverse Images of Regular Values

(Def 1) A subset S \subset R^{3} is a regular surface if, for each p \in S, there exists a neighborhood V in R^{3} and a map \mathbf{x} : U \to V \cap S of an open set U \subset R^{3} s.t.

\mathbf{x} is differentiable. This means that if we write \mathbf{x} (u,v) = (x(u,v), y(u,v), z(u,v)) , (u, v) \in U, the functions x(u,v) , y(u,v), z(u,v) have continuous partial derivatives of all orders in U.

\mathbf(x) is homeomorphism ; \mathbf{x} has an inverse \mathbf{x}^{-1} : V \cap S \to U which is continuous ; that is , \mathbf{x}^{-1} is the restriction of a continuous map F : W \subset R^{3} \to R^{2} defined on an open set W containing V \cap S.

(The regularity condition) For each q \in U, the differential d\mathbf{x}\_{q} : R^{2} \to R^{3} is one-to-one; d\mathbf{x}\_{q} = \begin{pmatrix} \frac{\partial x}{\partial u} &\frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}& \frac{\partial z}{\partial v} \end{pmatrix} \right|\_{q = (u\_{0}, v\_{0}) each column is linearly independent.

This mapping \mathbf{x} is called a parametrization or a system of (local) coordinates in (a neighborhood of) p. The neighborhood V \cap S of p in S is called a coordinate neighborhood.

(Prop 1) If f : U \to R is a differentiable function in an open set U of R^{2}. then the graph of f, that is , the subset of R^{3} given by (x,y,f(x,y)) for (x,y) \in U, is a regular surface.

(Def 2) Given a differential map F : U \subset R^{n} \to R^{m} defined in an open set U of R^{n} we say that p \in U is a critical point of F if the differential dF\_{p} : R^{n} \to R^{m} is not a surjective mapping. The image F(p) \in R^{m} of a critical point is called a critical value of F. A point of R^{m} which is not a critical value is called a regular value of F.

(prop 2) If f : U \subset R^{3} \to R is a differentiable function and a \in f(U) is a regular value of f, then f^{-1}(a) is a regular surface in R^{3}.

(prop 3) Let S \subset R^{3} be a regular surface and p \in S. Then there exists a neighborhood V of p in S s.t. V is the graph of a differentiable function which has one of the following three forms : z = f(x,y) , y = g(x,z), x = h(y,z).

(prop 4) Let p \in S be a point of a regular surface S and let \mathbf{x} : U \subset R^{2} \to R^{3} be a map with p \in \mathbf{x}(U) \subset S s.t. conditions 1 and 3 of (Def 1) holds. Assume that \mathbf{x} is one-to-one. then \mathbf{x}^{-1} is continuous.

(subsection 2.3.) Change of Parameters : Differentiable Function on Surfaces

(Change of Parameters)(Prop 1) Let p be a point of a regular surface S , and let \mathbf{X} : U \subset R^{2} \to S, y : V \subset R^{2} \to S be two parametrization of S s.t. p \in \mathbf{x}(U) \cap y(V) = W. Then the “Change of coordinates” h = \mathbf{x}^{-1} \bullet y : y^{-1}(W) \to \mathbf{x}^{-1}(W) is a diffeomorphism ; that is , h is differentiable and has a differentiable inverse h^{-1}.

(Def 1) Let f : V \subset S \to R be a function defined in an open subset V of a regular surface S. Then f is said to be differentiable at p \in V if, for some parametrization x : U \subset R^{2} \to S with p \in x(U) \subset V, the composition f \bullet x : U \subset R^{2} \to R is differentiable at x^{-1}(p) . f is differentiable in V if it is differentiable at all points of V.

(Def) Two regular surfaces S\_{1}, S\_{2} are diffeomorphic if there exists a differentiable map \phi : S\_{1} \to S\_{2} with a differentiable inverse \phi^{-1} : S\_{2} \to S\_{1} . such a \phi is called a diffeomorphism from S\_{1} to S\_{2}.

(Def 2) A parametrized surface x : U \subset R^{2} \to R^{3} is a differentiable map x from an open set U \subset R^{2} into R^{3}. The set x(U) \subset R^{3} is called the trace of x. x is regular if the differential dx\_{q} : R^{2} \to R^{3} is one-to-one for all q \in U (the vectors \partial x / \partial u , \partial x / \partial v are linearly independent for all q \in U) . A point p \in U where dx\_{q} is not one-to-one is called a singular point of x.

(Prop 2) Let x : U \subset R^{2} \to R^{3} be a regular parametrized surface and let q \in U. Then there exists a neighborhood V of q in R^{2} s.t. x(V) \subset R^{3} is a regular surface.

(subsection 2.4.) The tangent plane ; The differential of a map

(prop 1) Let x : U \subset R^{2} \to S be a parametrization of a regular surface S and let q \in U. The vector subspace of dimension 2, dx\_{q} (R^{2}) \subset R^{3} coincides with the set of tangent vectors to S at x(q).

The plane dx\_{q}(r^{2}) , which passes through x(q) = p , which does not depend on the parametrization x, will be called the tangent plane to S at p and will be denoted T\_{p}(S).

(Def) The coordinates of a vector w \in T\_{p}(S) in the basis associated to a parametrization x are determined as follows. w is the velocity vector \alpha’(0) of a curve \alpha = x \bullet \beta, where \beta : (-\epsilon, \epsilon) \to U is given by \beta(t) = (u(t), v(t)) , with \beta(0) = q = x^{-1}(p).

(prop2) In the discussion above, given w, the vector \beta ‘(0) does not depend on the choice of \alpha. The map d\phi\_{p} : T\_{p}(S\_{1}) \to T\_{\phi(p)}(S\_{2}) defined by d\phi\_{p}(w) = \beta’(0) is linear.

(Def) The linear map d\phi\_{p} defined by (prop2) is called the differential of \phi at p \in S\_{1}. In a similar way we define the differential of a (differentiable) function f : U \subset S \to R at p \in U as a linear map df\_{p} : T\_{p}(S) \to R.

A mapping \phi : U \subset S\_{1} \to S\_{2} is a local diffeomorphism at p \in U if there exists a neighborhood V \subset U of p s.t. \phi restricted to V is a diffeomorphism onto an open set \phi(V) \subset S\_{2}.

(prop 3) If S\_{1} and S\_{2} are regular surfaces and \phi : U \subset S\_{1} \to S\_{2} is a differentiable mapping of an open set U \subset S\_{1} s.t. the differential d\phi\_{p} of \phi at p \in U is an isomorphism, then \phi is a local diffeomorphism at p.

(subsection 2.5.) The First Fundamental form ; Area

(Def) The natural Inner product of R^{3} \supset S induces on each tangent plane T\_{p}(S) of a regular surface S an inner product, to be denoted by < , >\_{p} : If w\_{1}, w\_{2} \in T\_{p} (S) \subset R^{3} , then <w\_{1}, w\_{2}>\_{p} is equal to the inner product of w\_{1} and w\_{2} as vectors in R^{3}. There corresponds a quadratic form I\_{p} : T\_{p}(S) \to R given by I\_{p}(w) = <w,w>\_{p} = |w|^{2} \ge 0.

(Def 1) The quadratic form I\_{p} on T\_{q}(S), defined as above, is called the first fundamental form of the regular surface S \subset R^{3} at p \in S.

(Def) For a regular surface S, a (regular) domain of S is an open and connected subset of S s.t. its boundary is the image of a circle by a differentiable homeomorphism which is regular (that is, its differential is nonzero) except at a finite number of points. A region of S is the union of a domain with its boundary. A region of S \subset R^{3} is bounded if it is contained in some ball of R^{3}.

(Def 2) Let R \subset S be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization x : U \subset R^{2} \to S. The positive number \int \int\_{Q} |x\_{u} \wedge x\_{v}| du dv = A(R), Q = x^{-1}(R) is called the area of R.

(subsection 2.6.) Orientation of Surfaces

(Def 1) A regular surface S is called orientable if it is possible to cover it with a family of coordinate neighborhoods in such a way that if a point p \in S belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of such a family is called an orientation of S, and S, in this case, is called oriented. If such a choice is not possible, the surface is called nonorientable.

If a regular surface can be covered by two coordinate neighborhoods whose intersection is connected, then the surface is orentable.

(Prop1) A regular surface S \subset R^{3} is orientable iff there exists a differentiable field of unit normal vectors N : S \to R^{3} on S.

(prop 2) If a regular surface is given by S = {(x,y,z) \in R^{3} : f(x,y,z) = a}, where f : U \subset R^{3} \to R is differentiable and a is a regular value of f, then S is orientable.

(subsection 2.7.) A Characterization of Compact Orientable Surfaces

(Def) Let S \subset R^{3} be an orientable surface. On the normal line through p \in S, an open interval I\_{p} around p of length, say, 2 \epsilon\_{p} (\epsilon\_{p} varies with p) in such a way that if p \neq q \in S, then I\_{p} \cap I\_{q} = \empty. Thus, the union \bigcup I\_{p} , p \in S, constitutes an open set V of R^{3}, which contains S and has the property that through each point of V there passes a unique normal line to S; V is then called a tubular neighborhood of S.

(Prop 1) Let S be a regular surface and x : U \to S be a parametrization of a neighborhood of a point p = x(u\_{0}, v\_{0}) \in S. Then there exists a neighborhood W \subset x(U) of p in S and a number \epsilon >0 s.t. the segments of the normal lines passing through points q \in W, with center at q and length 2 \epsilon, are disjoint. (That is, W has a tubular neighborhood.)

(prop 2) Assume the existence of a tubular neighborhood V \subset R^{3} of an orientable surface S \subset R^{3}, and choose an orientation for S. Then the function g : V \to R , defined as the oriented distance form a point of V to the foot of the unique normal line passing through this point, is differentiable and has zero as a regular value.

(Bolzano-Weierstrass) (Prop 1) Let A \subset R^{3} be a compact set. Then every infinite subset of A has at least one limit point in A.

(Heine-Borel) (Prop 2) Let A \subset R^{3} be a compact set and {U\_{\alpha}} be a family of open sets of A s.t. \bigcup\_{\alpha} U\_{\alpha} = A. Then it is possible to choose a finite number U\_{k\_{1}}, U\_{k\_{2}}, …, U\_{k\_{n}} of U\_{\alpha} s.t. \bigcup U\_{k\_{i}} = A , i = 1,…,n.

(Lebesque) (prop 3) Let A \subset R^{3} be a compact set and {U\_{\alpha}} be a family of open sets of A s.t. \bigcup\_{\alpha} U\_{\alpha} = A. Then there exists a number \delta >0 (The Lebesgue number of the family {U\_{\alpha}}) s.t. whenever two points p,q \in A are at a distance d(p,q) < \delta then p and q belong to some U\_{\alpha}.

(Prop 3) Let S \subset R^{3} be a regular, compact, orientable surface. Then there exists a number \epsilon >0 s.t. whenever p, q \in S the segments of the normal lines of length 2 \epsilon , centered in p and q, are disjoint( that is, S has a tubular neighborhood).

(Thm) Let S \subset R^{3} be a regular compact orientable surface. Then there exists a differentiable function g : V \to R , defined in an open set V \subset R^{3}, with V \supset S (precisely a tubular neighborhood of S), which has zero as a regular value and is such that S = g^{-1}(0).

(subsection 2.8.) A geometric definition of Area

(Def) For a region R \subset S we define a partition \mathcal{P} of R into a finite number of regions R\_{i}, that is, we write R = \bigcup\_{i} R\_{i} , where intersection of two such regions R\_{i} is either empty or made up of boundary points of both regions. The diameter of R\_{i} is the supremum of the distances (in R^{3}) of any two points in R\_{i}. The largest diameter of the R\_{i}’s of a given partition \mathcal{P} is called the norm \mu of \mathcal{P}. If we take a partition of each R\_{i}, we obtain a second partition of R, which is said to refine \mathcal{P}.

Given a partition R = \bigcup\_{i} R\_{i} of R, we choose arbitrarily points p\_{i} \in R\_{i} and project R\_{i} onto the tangent palne at p\_{i} in the direction of normal line at p\_{i}; this projection is denoted by \bar{R} and its area b A(\bar{R}) .

If, by choosing partitions \mathcal{P}\_{1}, …, \mathcal{P}\_{n}, … more and more refined s.t. the norm \mu\_{n} of \mathcal{P}\_{n} converges to zero, there exists a limit of \sum\_{i} A(\bar{R}\_{i}) and this limit is independent of all choices. R has an area A(R) defined by A(R) = \lim\_{\mu\_{n} \to 0} \sum\_{i} A(\bar{R\_{i}}).

(Prop 1) Let x : U \to S be a coordinate system in a regular surface S and let R = x(Q) be a bounded region of S contained in x(U). Then R has an area given by A(R) = \int \int\_{Q} |x\_{u} \wedge x\_{v}| du dv.

(Appendix) A brief review of continuity and Differentiability

(Prop 1) F : U \subset R^{n} \to R^{m} is continuous iff each component function f\_{i} : U \subset R^{n} \to R , i = 1,…,m, is continuous.

(Prop 2) A map F : U \subset R^{n} \to R^{m} is continuous at p \in U iff , given a neighborhood V of F(p) in R^{m} there exists a neighborhood W of p in R^{n} s.t. F(W) \subset V.

(Prop 3) Let F : U \subset R^{n} \to R^{m} and G : V \subset R^{m} \to R^{k} be continuous maps, where U and V are open sets s.t. F(U) \subset V. Then G \bullet F : U \subset R^{n} \to R^{k} is a continuous map.

(Intermediate Value Theorem) (Prop 4) Let f : [a,b] \to R be a continuous function defined on the closed interval [a,b] . Assume that f(a) and f(b) have opposite signs; that is, f(a)f(b) < 0. Then there exists a point c \in (a,b) s.t. f(c) = 0.

(prop 5) Let f : [a,b] be a continuous function defined in the closed interval [a,b]. Then f reaches its maximum and its minimum in [a,b]; that is, there exists points x\_{1}, x\_{2} \in [a,b] s.t. f(x\_{1}) \le f(x) \le f(x\_{2}) for all x \in [a,b].

(Heine-Borel) (prop 6) Let [a,b] be a closed interval and let I\_{\alpha} , \alpha \in A be a collection of open intervals in [a,b] s.t. \bigcup\_{\alpha} I\_{\alpha} = [a,b] . Then it is possible to choose a finite number I\_{k\_{1}}, I\_{k\_{2}}, …, I\_{k\_{n}} of I\_{\alpha} s.t. \bigcup I\_{k\_{i}} = I, i = 1,…,n.

(subsection B) Differentiability in R^{n}

(Def 1) Let F : U \subset R^{n} \to R^{m} be a differentiable map. To each p \in U we associate a linear map dF\_{p} : R^{n} \to R^{m} which is called the differential of F at p and is defined as follows. Let w \in R^{n} and let \alpha : (- \epsilon, \epsilon) \to U be a differentiable curve s.t. \alpha(0) = p, \alpha’(0) = w. By the chain rule, the curve \beta = F \bullet \alpha : (-\epsilon , \epsilon) \to R^{m} is also differentiable. Then dF\_{p}(w) = \beta’(0) .

(prop 7.) The above definition of dF\_{p} does not depend on the choice of the curve which passes through p with tangent vector w, and dF\_{p} is , in fact, a linear map.

(The Chain rule for Maps) (Prop 8) Let F : U \subset R^{n} \to R^{m} and G : V \subset R^{m} \to R^{k} be differentiable maps, where U and V are open sets s.t. F(U) \subset V. Then G \bullet F : U \to R^{k} is a differentiable map, and d(G \bullet F)\_{p} = dG\_{F(p)} \bullet dF\_{p} , p \in U.

(Prop 9) Let f : U \subset R^{n} \to R be a differentiable function defined on a connected open subset U of R^{n}. Assume that df\_{p} : R^{n} \to R is zero at every point p \in U. Then f is constant on U.

(Inverse Function Theorem) Let F : U \subset R^{n} \to R^{n} be a differentiable mapping and suppose that at p \in U the differential dF\_{p} : R^{n} \to R^{n} is an isomorphism. Then there exists a neighborhood V of p in U and a neighborhood W of F(p) in R^{n} s.t. F:V \to W has a differentiable inverse F^{-1} : W \to V.

(Def) A differentiable mapping F : V \subset R^{n} \to W \subset R^{n}, where V and W are open sets, is called a diffeomorphism of V with W if F has a differentiable inverse.

(section 3) The Geometry of the Gauss Map

(subsection 3.2.) The Definition of the Gauss Map and its Fundamental properties

(Def) If V \subset S is an open set in S and N : V \to R^{3} is a differentiable map which associates to each q \in V a unit normal vector at q , we say that N is a differentiable field of unit normal vectors on V.

A regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field N is called an orientation of S.

(Def 1) Let S \subset R^{3} be a surface with an orientation N. The map N : S \to R^{3} takes its values in the unit sphere S^{2} = {(x,y,z) \in R^{3} ; | x^{2} + y^{2} + z^{2} = 1} The map N : S \to S^{2} , thus defined, is called the Gauss map of S.

(Prop 1) The differential dN\_{p} : T\_{p}(S) \to T\_{p} (S) of the Gauss map is a self-adjoint linear map.

(Def 2) The quadratic form II\_{p}, defined in T\_{p}(S) by II\_{p}(v) = - \langle dN\_{p}(v), v \rangle is called the second fundamental form of S at p.

(Def 3) Let C be a regular curve in S passing through p \in S, k the curvature of C at p, and cos \theat = \langle n, N \rangle, where n is the normal vector to C and N is the normal vector to S at p. The number k\_{n} = k cos \theta is then called the normal curvature of C \subset S at p.

(Meusnier) (prop 2) All curves lying on a surface S and having at a given point p \in S the same tangent line have at this point the same normal curvatures.

(Def 4) The maximum normal curvature k\_{1} and the minimum normal curvature k\_{2} are called the principal curvatures at p; the corresponding directions, that is, the directions given by the eigenvectors e\_{1}, e\_{2} are called principal directions at p.

(Def 5) If a regular connected curve C on S is such that for all p \in C the tangent line of C is a principal direction at p, then C is said to be a line of curvature of S.

(Olinde Rodrigues) (Prop 3) A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that N’(t) = \lambda(t) \alpha’(t) for any parametrization \alpha(t) of C, where N(t) = N \bullet \alpha(t) and \lambda (t) is a differentiable function on t. In this case, \lambda(t) is the (principal) curvature along \alpha’(t).

(Def 6) Let p \in S and let dN\_{p} : T\_{p}(S) \to T\_{p}(S) be the differential of the Gauss map. The determinant of dN\_{p} is the Gaussian curvature K of S at p. The negative of half of the trace of dN\_{p} is called the mean curvature H of S at p.

(Def 7) A point of a surface S is called

Elliptic if det(dN\_{p}) >0.

Hyperbolic if det(dN\_{p}) < 0.

Parabolic if det(dN\_{p}) = 0 , with dN\_{p} \neq 0.

Planar if dN\_{p} = 0.

(Def 8) If at p \in S, k\_{1} = k\_{2}, then p is called an umbilical point of S; in particular, the planar points (k\_{1} = k\_{2} = 0) are umbilical points.

(Prop 4) If all points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.

(Def 9) Let p be a point in S. An asymptotic direction of S at p is a direction of T\_{p}(S) for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve C \subset S such that for each p \in C the tangent line of C at p is an asymptotic direction.

(Def) Let p be a point in S. the Dupin indicatrix at p is the set of vectors w of T\_{p}(S) such that II\_{p}(w) = \mp 1.

(Def 10) Let p be a point on a surface S. Two nonzero vectors w\_{1}, w\_{2} \in T\_{p}(S) are conjugate if \langle dN\_{p}(w\_{1}) , w\_{2} \rangle = \langle w\_{1}, dN\_{p} (w\_{2}) \rangle = 0. Two directions r\_{1}, r\_{2} at p are conjugate if a pair of nonzero vectors w\_{1}, w\_{2} parallel to r\_{1} and r\_{2} , respectively, are conjugate.

(subsection 3.3.) The Gauss map in Local Coordinates

(Prop 1) Let p \in S be an elliptic point of a surface S. then there exists a neighborhood V of p in S s.t. all points in V belong to the same side of the tangent line T\_{p}(S). Let p \in S be a hyperbolic point. Then in each neighborhood of p there exists points of S in both sides of T\_{p}(S).

(Prop 2) Let p be a point of a surface S s.t. the Gaussian curvature K(p) \neq 0 , and let V be a connected neighborhood of p where K does not change sign. Then K(p) = \lim\_{A \to 0} \frac{A’}{A} where A is the area of a region B \subset V containing p , A’ is the area of the image of B by the Gauss map N : S \to S^{2}, and the limit is taken through a sequence of regions B\_{n} that converges to p, in the sense that any sphere around p contains all B\_{n} for n sufficiently large.

(subsection 3.4.)Vector fields

(Def) A vector field in an open set U \subset R^{2} is a map which assigns to each q \in U a vector w(q) \in R^{2} . The vector field w is said to be differentiable if writing q = (x,y) and w(q) = (a(x,y) , b(x,y)) , the functions a and b are differentiable functions in U.

(Thm 1) Let w be a vector field in an open set U \subset R^{2}. Given p \in U, there exists a trajectory \alpha : I \to U of w (\alpha’(t) = w(\alpha(t)), t \in I) with \alpha(0) = p. This trajectory is unique in the following sense : Any other trajectory \beta : J \to U with \beta(0) = p agrees with \alpha in I \cap J.

(Thm 2) Let w be a vector field in an open set U \subset R^{2}. For each p \in U there exists a neighborhood V \subset U of p, an interval I, and a mapping \alpha : V \times I \to U s.t.

for a fixed q \in V, the curve \alpha(q,t) , t \in I, is the trajectory of w passing through q; that is, \alpha(q,0) = q, \frac{\partial \alpha}{\partial t} (q,t) = w(\alpha(q,t)).

\alpha is differentiable.

(Lem) Let w be a vector field in an open set U \subset R^{2} and let p \in U be such that w(p) \neq 0. Then there exists a neighborhood W \subset U of p and a differentiable function f: W \to R s.t. f is constant along each trajectory of w and df\_{n} \new 0 for all q \in W.

(Def) A field of directions r in an open set U \subset R^{2} is a correspondence which assigns to each p \in U a line r(p) in R^{2} passing through p. r is said to be differentiable at p \in U if there exists a nonzero differentiable vector field w, defined in a neighborhood V \subset U of p, such that for each q \in V, w(q) \neq 0 is a basis of r(q); r is differentiable in U if it is differentiabl for every p \in U.

To each nonzero differentiable vector field w in U \subset R^{2}, there corresponds a differentiable field of directions given by r(p) = line generated by w(p), p \in U.

(Def 1) A vector field w in an open set U \subset S of a regular surface S is a correspondence which assigns to each p \in U a vector w(p) \in T\_{p}(S). The vector field w is differentiable at p \in U if, for some parametrization x(u,v) at p, the functions a(u,v) and b(u,v) given by w(p) = a(u,v)x\_{u} + b(u,v)x\_{v} are differentiable functions at p; it is clear that this definition does not depend on the choice of x.

(Thm) Let w\_{1} and w\_{2} be two vector fields in an open set U \subset S, which are linearly independent at some point p \in U. Then it is possible to parametrize a neighborhood V \subset U of p in such a way that for each q \in V the coordinate lines of this parametrization passing through q are tangent to the lines determined by w\_{1}(q) and w\_{2}(q).

(Cor 1) Given two fields of directions r and r’ in an open set U \subset S s.t. at p \in U, r(p) \neq r’(p), there exists a parametrization x in a neighborhood of p s.t. the coordinate curves of x are the integral curves of r and r’.

(Cor 2) For all p \in S there exists a parametrization x(u,v) in a neighborhood V of p s.t. the coordinate curves u = const., v = const. intersect orthogonally for each q \in V (such an x is called an orthogonal parametrization).

(Cor 3) Let p \in S be a hyperbolic point of S. Then it is possible to parametrize a neighborhood of p in such a way that the coordinate curves of this parametrization are the asymptotic curves of S.

(Cor 4) Let p \in S be a nonumbilical points of S. Then it is possible to parametrize a neighborhood of p in such a way that the coordinate curves of this parametrization are the lines of curvature of S.

(subsection 3.5.) Ruled Surfaces and Minimal Surfaces

(Subsubsection A) Ruled Surfaces

(Def) A (differentiable) one-parameter family of (straight) lines [\alpha(t), w(t)] is a correspondence that assigns to each t \in I a point \alpha(t) \in R^{3} and a vector w(t) \in R^{3} , w(t) \neq 0 , so that both \alpha(t) and w(t) depend differentiably on t. For each t \in I, the line L, which passes through \alpha(t) and is parallel to w(t) is called the line of the family at t.

Given a one-parameter family of lines [\alpha(t), w(t)] , the parametrized surface x(t,v) = \alpha(t) + vw(t) , t \in I, v \in R, is called the ruled surface generated by the family [\alpha(t), w(t)]. The lines are called the rullings, and the curve \alpha(t) is called a directrix of the surface x. Sometimes we use the expression ruled surface to mean the trace of x.

(subsubsection B)Minimal Surfaces

(Def) A regular parametrized surface is called minimal if its mean curvature vanishes everywhere. A regular surface S \subset R^{3} is minimal if each of its parametrization is minimal.

Let x : U \subset R^{2} \to R^{3} be a regular parametrized surface. Choose a bounded domain D \subset U and a differentiable function h : \bar{D} \to R, where \bar{D} is the union of the domain D with its boundary \partial D. The normal variation of x(\bar{D}) , determined by h, is the map given by, \phi : \bar{D} \times (\epsilon, \epsilon) \to R^{3} : \phi (u,v,t) = x(u,v) + th(u,v)N(u,v) , (u,v) \in \bar{D} , t \in (-\epsilon, \epsilon).

The mean curvature H is H = \frac{1}{2} \frac{Eg – 2fF + Ge}{EG-F^{2}} where \langle x\_{u}, N\_{u} \rangle = -e , \lange x\_{u} , N\_{v} \rangle + \langle x\_{v} , N\_{u} \rangle = -2f , \langle x\_{v} , N\_{v} \rangle = -g.

The derivative of the area A(t) of x’(\bar{D}) at t = 0 is A’(0) = -\int\_{\bar{D}}2hH \sqrt{EG-F^{2}} du dv. the area A(t) of x^{t}(\bar{D}) is A(t) = \int\_{\bar{D}} \sqrt{E^{t}G^{t} – (F^{t})^{2}} du dv

(Prop 1) Let x : U \to R^{3} be a regular parametrized surface and let D \subset U be a bounded domain in U. Then x is minimal iff A’(0) = 0 for all such D and all normal variations of x(\bar{D}).

(Def) For an arbitrary parametrized regular surface, the mean curvature vector defined by \mathbf{H} = HN. A regular parametrized surface x = x(u,v) is said to be isothermal if \langle x\_{u}, x\_{u} \rangle = \langle x\_{v} , x\_{v} \rangle and \langle x\_{u} , x\_{v} \rangle = 0.

(Prop 2) Let x = x (u,v) be a regular parametrized surface and assume that x is isothermal. Then x\_{uu} + x\_{vv} = 2 \lambda^{2}\mathbf{H}, where \lambda^{2} = \langle x\_{u}, x\_{u} \rangle = \langle x\_{v}, x\_{v} \rangle.

(Def) The Laplacial \Delta f of a differentiable function f : U \subset R^{2} \to R is defined by \Delta f = (\partial^{2}f / \partial u^{2}) + (\partial^{2}f / \partial v^{2}) , (u,v) \in U. We say that f is harmonic in U if \Delta f = 0 .

(Cor) Let \mathbf{x}(u,v) = (x(u,v) , y(u,v), z(u,v)) be a parametrized surface and assume that x is isothermal. Then x is minimal iff its coordinate functions x,y,z are harmonic.

(Def) A function f : U \subset \mathbb{C} \to \mathbb{C} is analytic when, f(\zeta) = f\_{1}(u,v) + i f\_{2}(u,v), the real functions f\_{1} and f\_{2} have continuous partial derivatives of first order which satisfy the Cauchy-Riemann equations : \frac{\partial f\_{1}}{\partial u} = \frac{\partial f\_{2}}{\partial v}, \frac{\partial f\_{1}}{\partial v} = -\frac{\partial f\_{2}}{\partial u}.

Let x : U \subset R^{2} \to R^{3} be a regular parametrized surface and define complex functions \phi\_{1}, \phi\_{2}, \phi\_{3} by \phi\_{1}(\zeta) = \frac{x}{u} – i \frac{x}{v} , \phi\_{2}(\zeta) = \frac{y}{u} – i \frac{y}{v} , \phi\_{3}(\zeta) = \frac{z}{u} – i \frac{z}{v} , where x,y, z are the component functions of x.

(Lem) x is isothermal iff \phi\_{1}^{2} + \phi\_{2}^{2} + \phi\_{3}^{2} \equiv 0. If this last condition is satisfied, x is minimal iff \phi\_{1}, \phi\_{2} and \phi\_{3} are analytic functions.

(Osserman) (Thm) Let S \subset R^{3} be a regular, closed (as a subset of R^{3}) minimal surface in R^{3} which is not a plane. Then the image of the Gauss map N : S \to S^{2} is dense in the sphere S^{2} (that is, arbitrarily close to any point of S^{2} there is a point of N(S) \subset S^{2}) .

(Appendix) Self-adjoint Linear maps and Quadratic forms

(Def) V is a vector space of dimension 2, with an inner product \langle , \rangle. A linear map A : V \to V is self-adjoint if \langle Av , w \rangle = \langle v, Aw \rangle for all v , w \in V.

To each self-adjoint linear map we associate a map B : V \times V \to R defined B(v,w) = \langle Av , w \rangle. It is bilinear, and B(v,w) = B(w,v) so B is a bilinear symmetric form in V.

To each Symmetric bilinear form B in V, there corresponds a quadratic form Q in V given by Q(v) = B(v,v) , v \in V .

(Lem) If the function Q(x,y) = ax^{2} + 2bxy + cy^{2} , restricted to the unit circle x^{2} + y^{2} , has a maximum at the point (1,0), then b = 0.

(Prop) Given a quadratic form Q in V , there exists an orthonormal basis [e\_{1}, e\_{2}] of V s.t. if v \in V is given by v = xe\_{1} + ye\_{2}, then Q(v) = \lambda\_{1} x^{2} + \lambda\_{2} y^{2}, where \lambda\_{1} and \lambda\_{2} are the maximum and minimum, respectively, of Q on the unit circle |v| = 1.

(Thm) Let A : V \to V be a self-adjoint linear map. Then there exists an orthonormal basis [e\_{1}, e\_{2}] of V s.t. A(e\_{1}) = \lambda\_{1} e\_{1} , A(e\_{2}) = \lambda\_{2} e\_{2}. (that is, e\_{1} and e\_{2} are eigenvectors , and \lambda\_{1}, \lambda\_{2} are eigenvalues of A). In the basis [e\_{1}, e\_{2}] , the matrix of A is clearly diagonal and the elements \lambda\_{1} , \lambda\_{2} , \lambda\_{1} \ge \lambda\_{2}, on the diagonal are the maximum and the minimum, respectively, of the quadratic form Q(v) = \langle Av, v \rangle on the unit circle of V.

(section 4) The Intrinsic Geometry of Surfaces

(subsection 4.2) Isometries ; Conformal maps

(Def 1) S and \bar{S} will always denote regular surfaces. A diffeomorphism \phi : S \to \bar{S} is an isometry if for all p \in S and all pairs w\_{1} , w\_{2} \in T\_{p}(S) we have \langle w\_{1}, w\_{2} \rangle\_{p} = \langle d\phi\_{p} (w\_{1}) , d\phi\_{p}( w\_{2}) \rangle\_{\phi(p)} , the surfaces S and \bar{S} are then said to be isometric.

(Def 2) A map \phi : V \to \bar{S} of a neighborhood V of p \in S is a local isometry at p if there exists a neighborhood \bar{C} of \phi(p) \in \bar{S} s.t. \phi : V \to \bar{V} is an isometry. If there exists a local isometry into \bar{S} at every p \in S, the surface S is said to be locally isometric to \bar{S} . S and \bar{S} are locally isometric if S is locally isometric to \bar{S} and \bar{S} is locally isometric to S.

(Prop 1) Assume the existence of parametrizations x : U \to S and \bar{x} : U \to \bar{S} s.t. E = \bar{E}, F = \bar{F}, G = \bar{G} in U. Then the map \phi = \bar{x} \bullet x^{-1} ; x(U) \to \bar{S} is a local isometry.

(Def 3) A diffeomorphism \phi : S \to \bar{S} is called a conformal map if for all p \in S and all v\_{1} , v\_{2} \in T\_{p}(S) we have \langel d\phi\_{p}(v\_{1}, d\phi\_{p}(v\_{2}) \rangle = \lambda^{2}(p) \langle v\_{1}, v\_{2} \rangle\_{p} , where \lambda^{2} is a nowhere-zero differentiable function on S; the surfaces S and \bar{S} are then said to be conformal. A map \phi : V \to \bar{S} of a neighbhorhood V of p \in S into \bar{S} is a local conformal map at p if there exists a neighborhood \bar{V} of \phi(p) s.t. \phi : V \to \bar{V} is a conformal map. If for each p \in S, there exists a local conformal map at p, the surface S is said to be locally conformal to \bar{S}.

(Prop 2) Let x : U \to S and \bar{x} : U \to \bar{S} be parametrizations s.t. E = \lambda^{2} \bar{E}, F = \lambda^{2} \bar{F}, G = \lambda^{2} \bar{G} in U, where \lambda^{2} is a nowhere-zero differentiable function in U. Then the map \phi = \bar{x} \bullet x^{-1} : x (U) \to \bar{S} is a local conformal map.

(Thm) Any two regular surfaces are locally conformal.

(subsection 4.3.) The Gauss Theorem and the Equations of Compatibility

(Def) Given x : U \subset R^{2} \to S be a parametrization in the orientation of S. By expressing the derivatives of the vectors x\_{u}, x\_{v} and N in the basis [x\_{u}, x\_{v} , N] , we obtain

x\_{uu} = \Gamma\_{1}^{1}\_{1} x\_{u} + \Gamma\_{1}^{2}\_{1} x\_{v} + L\_{1}N ,

x\_{uv} = \Gamma\_{1}^{1}\_{2} x\_{u} + \Gamma\_{1}^{2}\_{2} x\_{v} + L\_{2}N ,

x\_{vu} = \Gamma\_{2}^{1}\_{1} x\_{u} + \Gamma\_{2}^{2}\_{1} x\_{v} + L\_{3}N ,

x\_{vv} = \Gamma\_{2}^{1}\_{2} x\_{u} + \Gamma\_{2}^{2}\_{2} x\_{v} + L\_{4}N ,

N\_{u} = a\_{1 1} x\_{u} + a\_{21} x\_{v} ,

N\_{v} = a\_{12} x\_{u} + a\_{22} x\_{v}.

where the a\_{ij} , i,j = 1,2, were obtained in Chap. 3 and the other coefficients are to be determined. The coefficients \Gamma\_{ij}^{k} ,i,j,k = 1,2, are called the Christoffel symbols of S in the parametrization x. Since x\_{uv} = x\_{vu} , \Gamma\_{12}^{1} = \Gamma\_{21}^{1} and \Gamma\_{12}^{2} = \Gamma\_{21}^{2}.

We obtain L\_{1} = e , L\_{2} = \bar{L\_{2}} = f, L\_{3} = g, where e,f,g are coefficients of the second fundamental form of S.

All geometric concepts and properties expressed in terms of Christoffel symbols are invariant under isometries.

(Theorema Egregium) (Gauss) The Gaussian curvature K of a surface is invariant by local isometries.

(Def) Gauss formula : (\Gamma\_{12}^{2})\_{u} – (\Gamma\_{11}^{2})\_{v} + \Gamma\_{12}^{1}\Gamma\_{11}^{2} +\Gamma\_{12}^{2}\Gamma\_{12}^{2} - \Gamma\_{11}^{2}\Gamma\_{22}^{2} - \Gamma\_{11}^{1}\Gamma\_{12}^{2} = -E \frac{eg-f^{2}}{EG-F^{2}} = -EK.

(Def) Mainardi-Codazzi equations

e\_{v} – f\_{u} = e\Gamma\_{12}^{1} + f(\Gamma\_{12}^{2} - \Gamma\_{11}^{1}) – g \Gamma\_{11}^{2}

f\_{v} – g\_{u} = e \Gamma\_{22}^{1} + f(\Gamma\_{22}^{2} - \Gamma\_{12}^{1} ) -g \Gamma\_{12}^{2}

(Bonnet) (Thm) Let E,F,G,e,f,g, be differentiable functions, defined in an open set V \subset R^{2} , with E >0, and G >0. Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that EG-F^{2} >0. Then, for every q \in V there exists a neighborhood U \subset V of q and a diffeomorphism x : U \to x(U) \subset R^{3} s.t. the regular surface x(U) \subset R^{3} has E,F,G, and e,f,g, as coefficients of the first and second fundamental forms, respectively. Furthermore, if U is connected and if \bar{x} : U \to \bar{x}(U) \subset R^{3} is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation \rho in R^{3} s.t. \bar{x} = T \bullet \rho \bullet x.

(subsection 4.4.) Parallel Transport . Geodesics.

(Def 1) Let w be a differentiable vector field in an open set U \subset S and p \in U. Let y \in T\_{p}(S). Consider a parametrized curve \alpha : (-\epsilon , \epsilon) \to U, with \alpha(0) = p and \alpha’(0) = y, and let w(t) , t \in (- \epsilon, \epsilon) , be the restriction of the vector field w to the curve \alpha. The vector obtained by the normal projection of (dw/dt)(0) onto the plane T\_{p}(S) is called the covariant derivative at p of the vector field w relative to the vector y. This covariant derivative is denoted by (Dw/dt)(0) or (D,w)(p)

(Def 2) A parametrized curve \alpha : [0,l] \to S is the restriction to [0,l] of a differentiable mapping of (0 - \epsilon , l + \epsilon) , \epsilon >0, into S. If \alpha(0) = p and \alpha(l) = q, we say that \alpha joins p to q. \alpha is regular if \alpha’(t) \neq 0 for t \in [0, l].

(Def 3) Let \alpha : I \to S be a parametrized curve in S. A vector field w along \alpha is a correspondence that assigns to each t \in I a vector w(t) \in T\_{\alpha(t)} (S). The vector field w is differentiable at t\_{0} \in I if for some parametrization x(u,v) in \alpha(t\_{0}) the components a(t), b(t) of w(t) = ax\_{u} + bx\_{v} are differentiable functions of t at t\_{0}. w is differentiable in I if it is differentiable for every t \in I.

(Def 4) Let w be a differentiable vector field along \alpha : I \to S. The expression

\frac{Dw}{dt} = (a’ + \Gamma\_{11}^{1} au’ + \Gamma\_{12}^{1} av’ + \Gamma\_{12}^{1}bu’ + \Gamma\_{22}^{1}bv’) x\_{u} + (b’ + \Gamma\_{11}^{2}au’ + \Gamma\_{12}^{2} av’ + \Gamma\_{12}^{2}bu’ + \Gamma\_{22}^{2} bv’) x\_{v}

of (Dw/dt)(t), t \in I , is well-defined and is called the covariant derivative of w at t.

(Def 5) A vector field w along a parametrized curve \alpha : i \to S is said to be parallel if Dw/dt = 0 for every t \in I.

(prop 1) Let w and v be parallel vector fields along \alpha : I \to S. Then \langle w(t), v(t) \rangle is constant. In particular, |w(t)| and |v(t)| are constant, and the angle between v(t) and w(t) is constant.

(prop 2) Let \alpha : I \to S be a parametrized curve in S and let w\_{0} \in T\_{\alpha(t\_{0}) (S) , t\_{0} \in I. Then there exists a unique parallel vector field w(t) along \alpha(t) , with w(t\_{0}) = w\_{0}.

(Def 6) Let \alpha : I \to S be a parametrized curve and w\_{0} \in T\_{\alpha(t\_{0})} (S) , t\_{0} \in I. Let w be the parallel vector field along \alpha, with w(t\_{0}) = w\_{0}. The vector w(t\_{1}) , t\_{1} \in I, is called the parallel transport of w\_{0} along \alpha at the point t\_{1}.

(Def 7) A map \alpha : [0,l] \to S is a parametrized piecewise regular curve if \alpha is continuous and there exists a subdivision 0 = t\_{0} < t\_{1} < \cdots < t\_{k} < t\_{k+1} = l of the interval [0,l] in such a way that the restriction \alpha | [t\_{i}, t\_{i+1}] , i = 0, …, k is a parametrized regular curve. Each \alpha|[t\_{i},t\_{i+1}] is called a regular arc of \alpha .

(Def 8) A nonconstant, parametrized curve \gamma : I \to S is said to be geodesic at t \in I if the field of tangent vectors \gamma ‘(t) is parallel along \gamma at t; that is , \frac{D\gamma’(t)}{dt} = 0 ; \gamma is a parametrized geodesic if it is geodesic for all t \in I.

(Def 8a) A regular connected curve C in S is said to be a geodesic if , for every p \in S, the parametrization \alpha(s) of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, \alpha’(s) is a parallel vector field along \alpha(s).

(Def 9) Let w be a differentiable field of unit vectors along a parametrized curve \alpha : I \to S on an oriented surface S. Since w(t), t \in I, is a unit vector field, (dw/dt)(t) is a normal to w(t), and therefore \frac{Dw}{dt} = \lambda(N \wedge w(t)) . The real number \lambda = \lambda(T) , denoted by [Dw/dt] , is called the algebraic value of the covariant derivative of w at t.

(Def 10) Let C be an oriented regular curve contained on an oriented surface S ,and let \alpha(s) be a parametrization of C, in a neighborhood of p \in S, by the arc length s. The algebraic value of the covariant derivative [D\alpha’(s)/ds] = k\_{g} of \alpha’(s) at p is called the geodesic curvature of C at p.

(Lem 1) Let a and b be differentiable functions in I with a^{2} + b^{2} = 1 and \phi\_{0} be s.t. a(t\_{0}) = cos \phi\_{0} , b(t\_{0}) = sin \phi\_{0} . Then the differentiable function \phi = \phi\_{0} + \int\_{t\_{0}}^{t} (ab’ – ba’) dt is such that cos \phi(t) = a(t), sin \phi(t) = b(t) , t \in I, and \phi(t\_{0}) = \phi\_{0}.

(Lem 2) Let v and w be two differentiable vector fields along the curve \alpha : I \to S, with |w(t)| = |v(t)| = 1, t \in I, Then [\frac{Dw}{dt}] – [\frac{Dv}{dt}] = \frac{d\phi}{dt} where \phi is one of the differentiable determinations of the angle fron v to w, as given by (Lem 1).

(Prop 3) Let x(u,v) be an orthogonal parametrization (that is, F = 0) of a neighborhood of an oriented surface S, and w(t) be a differentiable field of unit vectors along the curve x(u(t), v(t)). Then [\frac{Dw}{dt}] = \frac{1}{2\sqrt{EG}} {G\_{u}\frac{dv}{dt} – E\_{v}\frac{du}{dt}} + \frac{d\phi}{dt} where \phi(t) is the angle from x\_{u} to w(t) in the given orientation.

(Liouville)(prop 4) Let \alpha(s) be a parametrization by arc length of a neighborhood of a point p \in S of a regular oriented curve C on an oriented surface S. Let x(u,v) be an orthogonal parametrization of S in p and \phi(s) be the angle that x\_{u} makes with \alpha’(s) in the given orientation. Then k\_{g} = (k\_{g})\_{1} cos \phi + (k\_{g})\_{2} sin \phi + \frac{d\phi}{ds} where (k\_{g})\_{1} and (k\_{g})\_{2} are the geodesic curvatures of the coordinate curves v = const. and u = const. respectively.

(Prop 5) Given a point p \in S and a vector w \in T\_{p}(S), w \neq 0 . There exists an \epsilon >0 and a unique parametrized geodesic \gamma : (-epsilon , \epsilon) \to S s.t. \gamma(0) = p , \gamma’(0) = w.

(subsection 4.5.) The Gauss-Bonnet Theorem and its Applications

(Def) Let \alpha : [0,l] \to S be a continuous map from the closed interval [0,l] into the regular surface S. We say that \alpha is a simple, closed, piecewise regular, parametrized curve if

\alpha(0) = \alpha(l).

t\_{1} \neq t\_{2}, t\_{1}, t\_{2} \in [0,l) implies that \alpha(t\_{1}) \neq \alpha(t\_{2}) .

There exists a subdivision 0 = t\_{0} < t\_{1} < \cdots < t\_{k} < t\_{k+1} = l of [0,l] s.t. \alpha is differentiable and regular in each [t\_{i}, t\_{i+1}] i = 0,…,k.

(Def) \alpha is a closed curve which fails to have a well-defined tangent line only at a finite number of points. The points \alpha(t\_{i}) , i = 0,…,k are called the vertices of \alpha and the traces \alpha([t\_{i}, t\_{i+1}]) are called the regular arcs of \alpha. Assume now that S is oriented and let |\theta\_{i}| , 0 < |\theta\_{i}| \le \pi , be the smallest determination of the angle from \alpha’(t\_{i} – 0) to \alpha’(t\_{i} + 0) . we give \theta\_{i} the sign of the determinant (\alpha’(t\_{i} – 0) , \alpha’(t\_{i} + 0), N).

(Def) Let x : U \subset R^{2} \to S be a parametrization compatible with the orientation of S. Assume that U is homeomorphic to an open disk in the plane.

Let \alpha : [0,l] \to x(U) \subset S be a simple closed, piecewise regular, parametrized curve, with vertices \alpha(t\_{i}) and external angles \theta\_{i}, i = 0,…,k.

Let \phi\_{i} : [t\_{i}, t\_{i+1}] \to R be differentiable functions which measure at each t \in [t\_{i}, t\_{i+1}] the positive angle from x\_{u} to \alpha’(t) (section 4-4 Lem 1)

(Thm)(Of Turning Tangents) With the above notation \sum\_{i = 0}^{k} (\phi\_{i}(t\_{i+1}) - \phi\_{i}(t\_{i})) + \sum\_{i = 0}^{k} \theta\_{i} = \mp 2\pi where the sign plus or minus depends on the orientation of \alpha.

(Def) Let x : U \subset R^{2} \to S be a parametrization of S compatible with its orientation and let R \subset x(U) be a bounded region of S. If f is a differentiable function on S, the integral \int \int\_{x^{-1}(R)} f(u,v) \sqrt{EG – F^{2}} du dv does not depend on the parametrization x, chosen in the class of orientation of x. It is called the integral of f over the region R. Denote it by \int \int\_{R} f d \sigma.

(Local)(Gauss Bonnet Theorem) Let x : U \to S be an orthogonal parametrization (that is , F = 0) , of an oriented surface S, where U \subset R^{2} is homeomorphic to an open disk and x is compatible with the orientation of S. Let R \subset x(U) be a simple region of S and let \alpha : I \to S be s.t. \partial R = \alpha(I). Assume that \alpha is positively oriented, parametrized by arc length s, and let \alpha(s\_{0}), \alpha(s\_{k}) and \theta\_{0} , …, \theta\_{k} be, respectively, the vertices and the external angles of \alpha. Then \sum\_{i = 0}^{k} \int\_{s\_{i}}^{s\_{i+1}} k\_{g}(s) ds + \int \int\_{R}K d \sigma + \sum\_{i = 0}^{k} \theta\_{o} = 2 \pi where k\_{g}(s) is the geodesic curvature of the regular arcs of \alpha and K is the Gaussian curvature of S.

(Def) Let S be a regular surface. A region R \subset S is said to be regular if R is compact and its boundary \partial R is the finite union of (simple) closed piecewise regular curves which do not intersect . We shall consider a compact surface as a regular region, the boundary of which is empty.

A simple region which has only three vertices with external angles \alpha\_{i} \neq 0 , i = 1,2,3, is called a triangle.

A triangulation of a regular region R \subset S is a finite family \mathfrak{J} of triangles T\_{i}, i = 1,….,n, s.t.

\bigcup\_{i=1}^{n} T\_{i} = R.

If T\_{i} \cap T\_{j} \neq \empty, then T\_{i} \cap T\_{j} is either a common edge of T\_{i} and T\_{j} or a common vertex of T\_{i} and T\_{j}.

Given a triangulation \mathfrak{J} of a regular region R \subset S of a surface S, denote F the number of triangles(faces), by E the number of sides(edges), by V the number of vertices of the triangulation. The number F-E+V = \chi is called The Euler-Poincare characteristic of the triangulation.

(prop 1) Every regular region of a regular surface admits a triangulation.

(prop 2) Let S be an oriented surface and {x\_{\alpha}}, \alpha \in A, a family of parametrizations compatible with the orientation of S. Let R \subset S be a regular region of S. Then there is a triangulation \mathfrak{J} of R s.t. every triangle T \in \mathfrak{J} is contained in some coordinate neighborhood of the family {x\_{\alpha}} . Furthermore, If the boundary of every triangle of \mathfrak{J} is positively oriented , adjacent triangles determine opposite orientations in the common edge.

(Prop 3) If R \subset S is a regular region of a surface S, the Euler-Poincare characteristic does not depend on the triangulation of R. It is convenient, therefore, to denote it \chi(R).

(Prop 4) Let S \subset R^{3} be a compact connected surface; then one of the values 2,0,-2,…,-2n, …, is assumed by the Euler-Poincare characteristic \chi(S). Furthermore, if S’ \subset R^{3} is another compact surface and \chi(S) = \chi(S’), then S is homeomorphic to S’.

(Prop 5) Let R \subset S be a regular region of an oriented surface S and let \mathfrak{J} be a triangulation of R s.t. every triangle T\_{j} \in \mathfrak{J} , j = 1,…,k, is contained in a coordinate neighborhood x\_{j}(U\_{j}) of a family of parametrizations {x\_{\alpha}} , \alpha \in A, compatible with the orientation of S. Let f be a differentiable function on S. the sum \sum\_{j = 1}^{k} \int \int\_{x\_{j}^{-1}(T\_{j}) f(u\_{j}, v\_{j}) \sqrt{E\_{j}G\_{j} – F\_{j}^{2}} du\_{j} dv\_{j} does not depend on the triangulation \mathfrak{J} or on the family {x\_{j}} of parametrizations of S.

(Global Gauss-Bonnet Thm) Let R \subset S be a regular region of an oriented surface and let C\_{1}, …, C\_{n} be the closed, simple, piecewise regular curves which form the boundary \partial R of R. Suppose that each C\_{i} is positively oriented and let \theta\_{1}, …, \theta\_{p} be the set of all external angles of the curves C\_{1}, …, C\_{n} . Then \sum\_{i = 1}^{n} \int\_{C\_{i}} k\_{g}(s) ds + \int \int\_{R} K d \sigma + \sum\_{i = 1}^{p} \theta\_{i} = 2 \pi \chi(R), where s denotes the arc length of C\_{i}, and the integral over C\_{i} means the sum of integrals in every regular arc of C\_{i}.

(Cor 1) If R is a simple region of S, then \sum\_{i = 0}^{k} \int\_{s\_{i}}^{s\_{i+1}} k\_{g}(s) ds + \int \int\_{R} k d \sigma + \sum\_{i = 0}^{k} \theta\_{i} = 2\pi.

(Cor 2) Let S be an orientable compact surface ; then \int \int\_{S} K d \sigma = 2 \pi \chi(S).

(Prop) Assume Every piecewise regular curve in the plane(thus without self-intersections) is the boundary of a simple region.

A compact surface of positive curvature is homeomorphic to a sphere.

Let S be an orientable surface of negative or zero curvature. Then two geodesics \gamma\_{1} and \gamma\_{2} which start from a point p \in S cannot mmet again at a point q \in S in such a way that the traces of \gamma\_{1} and \gamma\_{2} constitute the boundary of a simple region R of S.

Let S be a surface homeomorphic to a cylinder with Gaussian curvature K < 0. Then S has at most one simple closed geodesic.

If there exist two simple closed geodesics \Gamma\_{1} and \Gamma\_{2} on a compact surface S of a positive curvature, then \Gamma\_{1} and \Gamma\_{2} intetsect.

Let \alpha : I \to R^{3} be a closed, regular, parametrized curve with nonzero curvature. Assume that the curve described by normal vector n(s) in the unit sphere S^{2} (the normal indicatrix) is simple. Then n(I) divides S^{2} in two regions with equal areas.

Let T be a geodesic triangle(that is, the sides of T are geodesics) in an oriented surface S. Let \theta\_{1}, \theta\_{2}, \theta\_{3} be the external angles of T and let \phi\_{1} = \pi - \theta\_{1}, \phi\_{2} = \pi -\theta\_{2}, \phi\_{3} = \pi - \theta\_{3} be its interior angles. \int \int\_{T} K d \sigma = - \pi + \sum\_{i = 1}^{3} \phi\_{i}. The sum of the interior angles , \sum\_{i = 1}^{3} \phi\_{i} of a geodesic triangle is Equal to \phi if K = 0, Greater than \pi if K > 0 , Smaller that \pi if K < 0.

Furthermore, the difference - \pi + \sum\_{i = 1}^{3} \phi\_{i} (the excess of T) is given precisely by \int \int\_{T} K d \sigma. If K \neq 0 on T, this is the area of the image N(T) of T by the Gauss map N : S \to S^{2}. In other words, The excess of a geodesic triangle T is equal to the area of its spherical image N(T).

Vector fields on surfaces. Let v be a differentiable vector field on an oriented surface S. We say that p \in S is a singular point of v if v(p) = 0. The singular point p is isolated if there exists a neighborhood V of p in S such that v has no singular points in V other than p. let x : U \to S be an orthogonal parametrization at p = x(0,0) compatible with the orientation of S, and let \alpha : [0,l] \to S be a simple, closed, piecewise regular parametrized curve s.t. \alpha([0,l]) \subset x(U) is the boundary of a simple region R containing p as its only singular point. Let v = v(t), t \in [0,l] , be the restriction of v along \alpha, and let \phi = \phi(t) be some differentiable determination of the angle from x\_{u} to v(t), given by (Lem 1 of Sec 4.4) Since \alpha is closed, there is an integer I defined by 2\pi I = \phi(l) - \phi(0) = \int\_{0}^{t} \frac{d\phi}{dt} dt . I is called the index of v at p. (Poincare’s Theorem) The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler-Poincare characteristic of S.

(subsection 4.6.) The exponential map. Geodesic Polar Coordinates

(Lem 1) If the geodesic \gamma(t,v) is defined for t \in (-\epsilon , \epsilon) , then the geodesic \gamma (t, \lambda v) , \lambda \in R, \lambda \neq 0, is defined for t \in (-\epsilon / \lambda , \epsilon / \lambda) , and \gamma(t, \lambda v) = \gamma(\lambda t , v) .

(Def) If v = T\_{p}(S), v \neq 0, is s.t. \gamma(|v| , v/|v|) = \gamma(1,v) is defined, we set exp\_{p}(v) = \gamma(1,v) and exp\_{p}(0) = p.

(prop 1) Given p \in S there exists an \epsilon >0 s.t. exp\_{p} is defined and differentiable in the interior B\_{e} of a disk of radius \epsilon of T\_{p}(S), with center in the origin.

(prop 2) exp\_{p} : B\_{e} \subset T\_{p}(S) \to S is a diffeomorphism in a neighborhood U \subset B , of the origin 0 of T\_{p}(S).

(Def) V \subset S a normal neighborhood of p \in S if V is the image V = exp\_{p}(U) of a neighborhood U of the origin of T\_{p}(S) restricted to which exp\_{p} is a diffeomorphism.

Since the exponential map at p \in S is a diffeomorphism on U, it may be used to introduce coordinates in V. The normal coordinates which correspond to a system of rectangular coordinates in the tangent plane T\_{p}(S) . The geodesic polar coordinates which correspond to polar coordinates in the tangent plane T\_{p}(S).

polar coordinates in the plane are not defined in the closed half-line l which corresponds to \theta = 0. Set exp\_{p}(l) = L . Since exp\_{p} : U-l \to V-L is still a diffeomorphism, we may parametrize the points of V -L by the coordinates (\rho , \theta), which are called geodesic polar coordinates.

The images by exp\_{p} : U \to V of circles in U centered in 0 will be called geodesic circles of V, and the imagnes of exp\_{p} of the lines through 0 will be called radial geodesics of V.

(prop 3) Let x : U -l \to V-L be a system of geodesic polar coordinates (\rho , \theta). Then the coefficients E = E(\rho, \theta) , F = F(\rho, \theta) , and G = G(\rho, \theta) of the first fundamental form satisfy the conditions E = 1, F = 0, \lim\_{\rho \to 0} G = 0, \lim\_{\rho \to 0} (\sqrt(G))\_{\rho} = 1.

(Minding)(Thm) Any two regular surfaces with the same constant Gaussian curvature are locally isometric. More precisely, Let S\_{1}, S\_{2} be two regular surfaces with the same constant curvature K. Choose points p\_{1} \in S\_{1} , p\_{2} \in S\_{2} , and orthonormal basis {e\_{1}, e\_{2}} \in T\_{p\_{1}} (S\_{1}) , [f\_{1}, f\_{2}] \in T\_{p\_{2}} (S\_{2}) . Then there exists neighborhoods V\_{1} of p\_{1} , V\_{2} of p\_{2} and an isometry \psi : V\_{1} \to V\_{2} s.t. d\psi(e\_{1}) = f\_{1}, d\psi(e\_{2}) = f\_{2}.

(prop 4) Let p be a point on a surface S. Then, there exists a neighborhood W \subset S of p s.t. if \gamma : I \to W is a parametrized geodesic with \gamma(0)= p, \gamma(t\_{1}) = q, t\_{1} \in I, and \alpha : [0,t\_{1}] \to S is a parametrized regular curve joining p to q, we have l\_{\gamma} \le l\_{\alpha} , where I\_{\alpha} denotes the length of the curve \alpha. Moreofer, if I\_{\gamma} = I\_{\alpha} , then the trace of \alpha coincides with the trace of \alpha between p and q.

(Prop 5) Let \alpha : I \to S be a regular parametrized curve with a parameter proportional to arc length. Suppose that the arc length of \alpha between any two points t, \tau \in I, is smaller than or equal to the arc length of any regular parametrized curve joining \alpha(t) to \alpha(\tau) . Then \alpha is a geodesic.

(Thm 1) Let S a regular surface. Given p \in S there exist numbers \epsilon\_{1} >0 , \epsilon\_{2} >0 and a differentiable map \gamma : (-\epsilon\_{2}, \epsilon\_{2}) \times B\_{\epsilon\_{1}} \to S, B\_{\epsilon\_{1}} \subset T\_{p}(S) s.t. for v \in B\_{\epsilon\_{1}} , v \neq 0 , t \in (-\epsilon\_{2}, \epsilon\_{2}) the curve t \in \gamma(t,v) is the geodesic of S with \gamma(0,v) = p , \gamma’(0,v) = v and for v = 0 \gamma(t,0) = p.

(Thm 1a.) Given p \in S, there exists positive numbers \epsilon, \epsilon\_{1}, \epsilon\_{2} and a differentiable map \gamma : (-\epsilon\_{2}, \epsilon\_{2}) \times \mathcal{U} \to S where \mathcal{U} = {(q,v) ; q \in B\_{\epsilon}(p), v \in B\_{\epsilon\_{1}}(0) \subset T\_{q}(S)}, s.t. \gamm(t,q,0) =q, and for v \neq 0 the curve t \to \gamma(t,q,v), t \in (-\epsilon\_{2}, \epsilon\_{2}) is the geodesic of S with \gamma(0,q,v) = q, \gamma’(0,q,v) = v.

(prop 1) Given p \in S there exist a neighborhood W of p in S and a number \delta >0 s.t. for every q \in W, exp\_{q} is a diffeomorphism on B\_{\delta}(0) \subset T\_{q}(S) and exp\_{q}(B\_{\delta}(0)) \supset W ; that is, W is a normal neighborhood of all its points.

(prop 2) Let \alpha : I \to S be a parametrized, piecewise regular curve s.t. in each regular arc the parameter is proportional to the arc length. Suppose that the arc length between any two of this points is smaller than or equal to the arc length of any parametrized regular curve joining these points. Then \alpha is a geodesic ; in particular, \alpha is regular everywhere.

(prop 3) For each point p \in S there exists a positive number \epsilon with the following property ; If a geodesic \gamma(t) is tangent to the geodesic circle S\_{r}(p), r < \epsilon , at \gamma(0) , then for t \neq 0 small, \gamma(t) is outside B\_{t}(p) .

(Existence of Convex neighborhoods) (prop 4) For each point p \in S there exists a number c > 0 s.t. B\_{c}(p) is convex ; that is, any two points of B\_{c}(p) can be joined by a unique minimal geodesic in B\_{c}(p) .

(Section 5) Global Differential Geometry

(subsection 5.2.) The Rigidity of the Sphere

(Thm 1) Let S be a compact , connected, regular surface with constant Gaussian curvature K. Then S is a sphere.

(Lem 1) Let S be a regular surface and p \in S a point of S satisfying the following conditions :

K(p) >0 ; that is, the Gaussian curvature in p is positive.

p is simultaneously a point of local maximum for the function k\_{1} an a point of local minimum for the function k\_{2} (k\_{1} \ge k\_{2}) .

Then p is an umbilical point of S.

(Thm 1a.) Let S be a regular, compact, and connected surface with Gaussian curvature K > 0 and mean curvature H constant. Then S is a sphere.

(Thm 1b) Let S be a regular, compact, and connected surface of positive Gaussian curvature. If there exists a relation k\_{2} = f(k\_{1}) in S, where f is a decreasing function of k\_{1}, k\_{1} \ge k\_{2}, then S is a sphere.

(Lem 2) A regular compact surface S \subset R^{3} has at least one elliptic point.

(subsection 5.3.) Complete Surfaces . Theorem of Hopf-Rinow

(Def 1) A regular (connected) surface S is said to be extendable if there exists a regular (connected) surface \bar{S} s.t. S \subset \bar{S} as a proper subset. If there exists no such \bar{S}, S said to be nonextendable.

(Def 2) A regular surface S is said to be complete when for every point p \in S, any parametrized geodesic \gamma : [0, \epsilon) \to S of S, starting from p = \gamma(0) , may be extended into a parametrized geodesic \bar{\gamma} : R \to S, defined on the entire line R. In other words, S is complete when for every p \in S the mapping exp\_{p} : T\_{p} (S) \to S is defined for every v \in T\_{p}(S).

(Prop 1) A complete surface S is nonextendable.

(Prop 2) Given two points p, q \in S of a regular(connected) surface S, there exists a parametrized piecewise differentiable curve joining p to q.

(Def 3) The (intrinsic) distance d(p,q) from the point p \in S to the point q \in S is the number d(p,q) = \inf 1(\alpha\_{p,q}) , where the inf is taken over all piecewise differentiable curves joining p to q.

(Prop 3) The distance d defined above has the following properties.

d(p,q) = d(q,p),

d(p,q) + d(q,r) \ge d(p,r) ,

d(p,q) \ge 0,

d(p,q) = 0 iff p = q.

(Cor) |d(p,r) – d(r,q)| \le d(p,q).

(prop 4) If we let p\_{0} \in S be a point of S, then the function f : S \to R given by f(p) = d(p\_{0}, p) , p \in S, is continuous on S.

(prop 5) A closed surface S \subset R^{3} is complete.

(Cor) A compact surface is complete.

(Def) A geodesic \gamma joining two points p, q \in S is minimal if its length l(\gamma) is smaller than or equal to the length of any piecewise regular curve joining p to q.

(Hopf-Rinow) (Thm) Let S be a complete surface. Given two points p, q \in S, there exists a minimal geodesic joining p to q.

(Cor 1) Let S be complete. Then for every point p \in S the map exp\_{p} : T\_{p}(S) \to S is onto S.

(Cor 2) Let S be complete and bounded in the metric d (that is, there exists r >0 s.t. d(p,q) < r for every pair p,q \in S). Then S is compact.

(subsection 5.4.) First and Second Variations of Arc length ; Bonnet’s Theorem

(Def 1) Let \alpha : [0,l] \to S be a regular parametrized curve, where the parameter s \in [0,l] is the arc length. A variation of \alpha is a differentiable map h : [0,l] \times (-\epsilon, \epsilon) \subset R^{2} \to S s.t. h(s,0) = \alpha(s), s \in [0,l] . For each t \in (-\epsilon, \epsilon) , the curve h\_{t} : [0,l] \to S, given by h\_{t}(s) = h(s,t), is called a curve of the variation h. A variation h is said to be proper if h(0,t) = \alpha(0) , h(l,t) = \alpha(l), t \in (-\epsilon, \epsilon).

(Def) A variation h of \alpha determines a differentiable vector field V(s) along \alpha by V(s) = \frac{\partial h}{\partial s}(s,0) , s \in [0,l] . V is called the variational vector field of h; If h is proper, then V(0) = V(l) = 0.

(prop 1) If we let V(s) be a differentiable vector field along a parametrized regular curve \alpha : [0,l] \to S then there exists a variation h : [0,l] \times (-\epsilon , \epsilon) \to S of \alpha s.t. V(s) is the variational vector field of h. Furthermore, if V(0) = V(l) = 0, then h can be chosen to be proper.

(Def) Define a function L : (-\epsilon, \epsilon) \to R by L(t) = \int\_{0}^{l} |\frac{\partial h}{\partial s}(s,t)|ds, t \in (-\epsilon, \epsilon) .

(Lem 1) The function L defined above is differentiable in a neighborhood of t= 0 ; in such a neighborhood , the derivative of L may be obtained by differentiation under the integral sign.

(Lem 2) Let w(t) be a differentiable vector field along the parametrized curve \alpha : [a,b] \to S and let f : [a,b] \to R be a differentiable function. Then \frac{D}{dt}(f(t)w(t)) = f(t) \frac{Dw}{dt} + \frac{df}{dt}w(t).

(Lem 3) Let v(t) and w(t) be differentiable vector fields along the parametrized curve \alpha : [a,b] \to S. Then \frac{d}{dt} \langle v(t), w(t) \rangle = \langle \frac{Dv}{dt},w(t) \rangle + \langle v(t), \frac{Dw}{dt} \rangle.

(Lem 4) Let h : [0,l] \times (-\epsilon, \epsilon) \subset R^{2} \to S be a differentiable mapping. Then \frac{D}{\partial s} \frac{\partial h}{\partial t} (s,t) = \frac{D}{\partial t} \frac{\partial h}{\partial s} (s,t).

(prop 2) Let h : [0,l] \times (-\epsilon, \epsilon) be a proper variation of the curve \alpha : [0,l] \to S and let V(s) = (\partial h / \partial t)(s,0) , s \in [0,l] , be the variational vector field of h. Then L’(0) = -\int\_{0}^{l} \langle A(s) , V(s) \rangle ds, where A(s) = (D/\partial s) (\partial h / \partial s) (s,0).

(prop 3) A regular parametrized curve \alpha : [0,l] \to S , where the parameter s \in [0,l] is the arc length of \alpha , is a geodesic iff , for every proper variation h : [0,l] \times (-\epsilon , \epsilon) \to S of \alpha , L’(0) = 0.

(Lemma 5) Let x : U \to S be a parametrization at a point p \in S of a regular surface S, with parameters u,v, and let K be a Gaussian curvature of S. Then \frac{D}{\partial v}\frac{D}{\partial u} x\_{u} - \frac{D}{\partial u}\frac{D}{\partial v}x\_{u} = K(x\_{u} \wedge x\_{v}) \wedge x\_{u}.

(Lem 6) Let h : [0,l] \times (-\epsilon, \epsilon) \to S be a differentiable mapping and let V(s,t) , (s,t) \in [0,l] \times (-\epsilon , \epsilon) , be a differentiable vector field along h. Then \frac{D}{\partial t}\frac{D}{\partial s} V - \frac{D}{\partial s}\frac{D}{\partial t}V = K(s,t)(\frac{\partial h}{\partial s} \wedge \frac{\partial h}{\partial t}) \wedge V, where K(s,t) is the curvature of S at the point h(s,t).

(Prop 4) Let h : [0,l] \times (-\epsilon , \epsilon) \to S be a proper orthogonal variation of a geodesic \gamma :[0,l] \to S parametrized by the arc length s \in [0,l] . Let V(s) = (\partial h/\partial t) (s,0) be the variational vector field of h. Then L’’(0) = \int\_{0}^{t} (|\frac{D}{\partial s}V(s)|^{2} -K(s)|V(s)|^{2})ds where K(s) = K(s,0) is the Gaussian curvature of S at \gamma(s) = h(s,0).

(Bonnet)(Thm) Let the Gaussian curvature K of a complete surface S satisfy the condition K \ge \delta > 0 . Then S is compact and the diameter \rho of S satisfies the inequality \rho \le \frac{\pi}{\sqrt{\delta}} .

(subsection 5.5.) Jacobi Fields and Conjugate Points

(Def 1) Let \gamma : [0,l] \to S be a parametrized geodesic on S and let h : [0,l] \times (-\epsilon , \epsilon) \to S be a variation of \gamma s.t. for every t \in (-\epsilon , \epsilon) the curve h\_{t}(s) = h(s,t) , s \in [0,l] , is a parametrized geodesic (not necessarily parametrized by arc length) . The variational field (\partial h/\partial h)(s,0) = J(s) is called a Jacobi field along \gamma.

(Prop 1) Let J(s) be a Jacobi field along \gamma : [0,l] \to S, s \in [0,l] . Then J satisfies the so-called Jacobi equation \frac{D}{ds}\frac{D}{ds}J(s) + K(s)(\gamma’(s) \wedge J(s)) \wedge \gamma’(s) = 0, where K(s) is the Gaussian curvature of S at \gamma(s).

(Lem 1) Let p \in S and choose v,w \in T\_{p}(S), with |v| = l. Let \gamma:[0,l] \to S be the geodesic on S given by \gamma(s) = exp\_{p}(sv) , s \in [0,l]. Then, the vector field J(s) along \gamma given by J(s) = s(d exp\_{p})\_{sv}(w) , s \in [0,l] , is a Jacobi field. Furthermore, J(0) = 0, (DJ/ds) (0) = w.

(Prop 2) If we let J(s) be a differentiable vector field along \gamma : [0,l] \to S, s \in [0,l] , satisfying the Jacobi equation with J(0) = 0, then J(s) is a Jacobi field along \gamma.

(Def 2) Let \gamma : [0,l] \to S be a geodesic of S with \gamma(0) = p. We say that the point q = \gamma(s\_{0}) , s\_{0} \in [0,l] , is conjugate to p relative to the geodesic \gamma (if there exists a Jacobi field J(s) which is not identically zero along \gamma with J(0) = J(s\_{0}) = J(s\_{0}) = 0.

(Prop 3) Let J\_{1}(s) and J\_{2}(s) be the Jacobi fields along \gamma : [0,l] \to S, s \in [0,l] . Then \langle \frac{DJ\_{1}}{ds}, J\_{2}(s) \rangle - \langle J\_{1}(s) , \frac{DJ\_{2}}{ds} \rangle = const.

(Cor) Let J(s) be a Jacobi field along \gamma : [0,l] \to S, with J(0) = J(l) = 0. Then \langle J(s) , \gamma’(s) \rangle = 0, s \in [0,l].

(Prop 5) Let p , q \in S be two points of S and let \gamma : [0,l] \to S be a geodesic joining p = \gamma(0) to q = exp\_{p}(l\gamma’(0)) . Then q is conjugate to p relative to \gamma iff v = l\gamma’(0) is a critical point of exp\_{p} : T\_{p}(S) \to S.

(Thm) Assume that the Gaussian curvature K of a surface S satisfies the condition K \le 0. Then, for every p \in S. the conjugate locus of p is empty. In short, a surface of curvature K \le 0 does not have conjugate points.

(Cor) Assume the Gaussian curvature K of S to be negative or zero. Then for every p \in S, the mapping exp\_{p} : T\_{p}(S) \to S is a local diffeomorphism.

(Gauss) (Lem 2) Let p \in S be a point of a (complete) surface S and let u \in T\_{p}(S) and w \in (T\_{p}(S))\_{u} . Then \langle u, w \rangle = \langle (d exp\_{p})\_{u}(u) , (d exp\_{p})\_{u}(w) \rangle , where the identification T\_{p}(S) \approx (T\_{p}(S))\_{u} is being used.

(subsection 5.6.) Covering Spaces ; The Theorem of Hadamard

(subsubsection A) Covering Spaces

(Def 1) Let \tilde{B} and B be subsets of R^{3} . We say that \pi : \tilde{B} \to B is a covering map if \pi is continuous and \pi(\tilde{B}) = B.

Each point p \in B has a neighborhood U in B (to be called a distinguished neighborhood of p) s.t. \pi^{-1}(U) = \bigcup\_{\alpha}V\_{\alpha}, where the V\_{\alpha}’s are pairwise disjoint open sets s.t. the restriction of \pi to V\_{\pi} is a homeomorphism of V\_{\alpha} onto U. \tilde{B} is then called a covering space of B.

(Prop 1) Let \pi : \tilde{B} \to B be a local homeomorphism, \tilde{B} compact and B connected. Then \pi is a covering map.

(Prop 2) Let \pi : \tilde{B} \to B be a coveing map, \alpha : [0,l] \to B an arc in B and \tilde{p}\_{0} \in \tilde{B} a point of \tilde{B} s.t. \pi(\tilde{p}\_{0}) = \alpha(0) = p\_{0} . Then there exists a unique lifting \tilde{\alpha} : [0,l] \to \tilde{B} with origin at \tilde{p}\_{0} , that is, with \tilde{\alpha}(0) = \tilde{p}\_{0}.

(Def 2) Let B \subset R^{3} and let \alpha\_{0} : [0,l] \to B, \alpha\_{1} : [0,l] \to B be two arcs of B, joining the points p = \alpha\_{0}(0) = \alpha\_{1}(0) , q = \alpha\_{0}(l)= \alpha\_{1}(l). We say that \alpha\_{0} and \alpha\_{1} are homotopic if there exists a continuous map H : [0,l] \times [0,l] \to B s.t. H(s,0) = \alpha\_{0}(s), H(s,1) = \alpha\_{1}(s), s \in [0,l] . H(0,t) = p, H(l,t) = q, t \in [0,1]. The map H is called a homotopy between \alpha\_{0} and \alpha\_{1}.

(Def) Let \pi : \tilde{B} \to B be a continuous map and let \alpha\_{0} , \alpha\_{1} \to B be two arcs of B joining the points p and q. Let H : [0,l] \times [0,1] \to B be a homotopy between \alpha\_{0} and \alpha\_{1}. If there exists a continuous map \tilde{H} : [0,l] \times [0,1] \to \tilde{B} s.t. \pi \bullet \tilde{H} = H, we say that \tilde{H} is a lifting of the homotopy H, with origin at \tilde{H}(0,0) = \tilde{p} \in \tilde{B}.

(prop 3) Let \pi : \tilde{B} \to B be a local homeomorphism with the property of lifting arcs. Let \alpha\_{0} , \alpha\_{1} : [0,l] \to B be two arcs of B joining the points p and q. Let H : [0,l] \times [0,1] \to B be a homotopy between \alpha\_{0} and \alpha\_{1} , and let \tilde{p} \in \tilde{B} be a point of \tilde{B} s.t. \pi(\tilde{p}) = p. Then there exists a unique lifting \tilde{H} of H with origin at \tilde{p}.

(Prop 4) Let \pi : \tilde{B} \to B be a local homeomorphism with the property of lifting arcs. Let \alpha\_{0} , \alpha\_{1} : [0,l] \to B be two arcs of B joining the points p and q and choose \tilde{p} \in \tilde{B} s.t. \pi(\tilde{p}) = p. If \alpha\_{0} and \alpha\_{1} are homotopic, then the liftings \tilde{\alpha\_{0}} and \tilde{\alpha\_{1}} , of \alpha\_{0} and \alpha\_{1}, respectively, with origin \tilde{p} , are homotopic.

(Def 3) An arcwise connected set B \subset R^{3} is simply connected if given two points p, q \in B and two arcs \alpha\_{0} : [0,l] \to B, \alpha\_{1} : [0,l] \to B joining p to q, there exists a homotopy in B between \alpha\_{0} and \alpha\_{1} . In particular, any closed arc of B, \alpha:[0,l] \to B (closed means that \alpha(0) = \alpha(l) = p) , is homotopic to the constant arc \alpha(s) = p, s \in [0,l]

(Prop 5) Let \pi : \tilde{B} \to B be a local homeomorphism with the property of lifting arcs. Let \tilde{B} be arcwise connected and B simply connected. Then \pi is a homeomorphism.

(Cor) Let \pi : \tilde{B} \to B be a covering map, \tilde{B} arcwise connected, and B simply connected. Then \pi is a homeomorphism.

(Def 4) B is locally simply connected if any neighborhood of each point contains a simply connected neighborhood.

(Prop 6) Let \pi : \tilde{B} \to B be a local homeomorphism with the property of lifting arcs. Assume that B is locally simply connected and that \tilde{B} is locally arcwise connected. Then \pi is a covering map.

(subsubsection B) The Hadamard Theorem

(Lem 1) Let S be a complete surface of curvature K \le 0. Then exp\_{p} : T\_{p} (S) \to S , p \in S, is length-increasing in the following sense : If u, w \in T\_{p}(S), we have \langle(d exp\_{p})\_{u}(w), (d exp\_{p})\_{u}(w) \rangle \ge \langle w, w \rangle, where, as usual, w denotes a vector in (T\_{p}(S))\_{u} that is obtained from w by the translation u.

(Cor) Let K \equiv 0. Then exp\_{p} : T\_{p}(S) \to S, p \in S, is a local isometry.

(Prop 7) Let S be a complete surface with Gaussian curvature K \le 0. Then the map exp\_{p} : T\_{p}(S) \to S, p \in S, is a covering map.

(Hadamard)(Thm 1) Let S be a simply connected, complete surface, with Gaussian curvature K \le 0. Then exp\_{p} : T\_{p}(S) \to S, p \in S, is a diffeomorphism; that is , S is diffeomorphic to a plane.

(Hadamard) (Thm 2) Let S be an ovaloid(Connected, compact, regular surface with Gaussian curvature K > 0) . Then the Gauss map N : S \to S^{2} is a diffeomorphism. In particular, S is diffeomorphic to a sphere.

(subsection 5.7.) Global Theorems for curves; The Fary-Milnor Theorem

(Def) Let \phi : S^{1} \to S^{1} be a continuous map. \phi can be though as continuous map \phi : [0,l] \to S^{1}, with \phi(0) = \phi(l) = p \in S^{1}. Thus, \phi is a closed arc at p in S^{1} which, by (Prop 2) of (Sec 5-6) , can be lifted into a unique arc \tilde{\phi} : [0,l] \to R, starting at a point x \in R with \pi(x) = p. Since \pi(\tilde{\phi}(0)) = \pi(\tilde{\phi}), the difference \tilde{\phi}(l) - \tilde{\phi}(0) is an integral multiple of 2 \pi. The integer given by \tilde{\phi}(l) - \tilde{\phi}(0) = (deg \phi) 2\pi is called the degree of \phi.

(Differentiable Jordan Curve Theorem) (Thm 1) Let \alpha : [0,l] \to R^{2} be a plane, regular, closed, simple curve. Then R^{2} - \alpha([0,l]) has exactly two connected components, and \alpha([0,l]) is their common boundary.

(Def) A plane, regular, closed curve \alpha : [0,l] \to R^{2} is convex if, for each t \in [0,l] , the curve lies in one of the closed half-planes determined by the tangent line at t.

(Thm 2) Let \beta : [0,l] \to R^{2} be a plane, regular, simple, closed curve. Then the rotation index of \beta is \mp 1 (depending on the orientation of \beta)

(Prop 1) A plane, regular, closed curve is convex iff it is simple and its curvature k does not change sign.

(Fenchel’s Theorem) (Thm 3) The total curvature of a simple closed curve is \ge 2\pi, and equality holds iff the curve is a plane convex curve.

(Def) A simple closed continuous curve C \subset R^{3} is unknotted if there exists a homotopy H : S^{1} \times I \to R^{3} , I = [0,1], s.t. H(S^{1} \times {0}) = S^{1}, H(S^{1} \times {1}) = C; and H(S^{1} \times {t}) = C\_{t} \subset R^{3} is homeomorphic to S^{1} for all t \in [0,1]. Such a homotopy is called an isotopy; an unknotted curve is then a curve isotopic to S^{1}. When this is not the case, C is said to be knotted.

(Fary-Milnor)(Thm 4) The total curvature of a knotted simple closed curve is greater than 4\pi.

(subsection 5.8.) Surfaces of Zero Gaussian Curvature

(Def) A cylinder is a regular surface S s.t. through each point p \in S there passes a unique line R(p) \subset S (the generator through p) which satisfies the condition that if q \neq p, then the lines R(p) and R(q) are parallel or equal.

(Thm) Let S \subset R^{3} be a complete surface with zero Gaussian curvature. Then S is a cylinder or a plane.

(Prop 1) The unique asymptotic line that passes through a parabolic point p \in U \subset S of a surface S of curvature K \equiv 0 is an (open) segment of a (straight) line in S.

(Massey)(Prop 2) Let r be a maximal asymptotic line passing through a parabolic point p \in U \subset S of a surface S of curvature K \equiv 0 and let P \subset S be the set of planar points of S. Then r \cap P = \empty.

(Lem 1) Let s be the arc length of the asymptotic curve passing through a parabolic point p of a surface S of zero curvature and let H = H(s) be the mean curvature of S along this curve. Then , in U, \frac{d^{2}}{ds^{2}}(\frac{1}{H}) = 0.

(Def) Let now Bd(U) be the boundary of U in S ; that is, Bd(U) is the set of points p \in S s.t. every neighborhood of p in S contains points of U and points of S – U = P.

(Massey)(Prop 3) Let p \in Bd(U) \subset S be a point of the boundary of the set U of parabolic points of a surface S of curvature K \equiv 0. Then through p there passes a unique open segment of line C(p) \subset S . Furthermore, C(p) \subset Bd(U); that is, the boundary of U is formed by segments of lines.

(subsection 5.9.) Jacobi’s Theorems

(Lem 1) Let p \in S, u \in T\_{p}(S), l = |u| \neq 0, and let \bar{\gamma} : [0,l] \to T\_{p}(S) be the line of T\_{p}(S) given by \bar{\gamma}(s) = sv, s \in [0,l] , v = \frac{u}{|u|}. Let \bar{\alpha} : [0,l] \to T\_{p}(S) be a differentiable parametrized curve of T\_{p}(S), with \bar{\alpha} (0) = 0, \bar{\alpha}(l) = u, and \bar{\alpha}(s) \neq 0 if s \neq 0. Furthermore, let \alpha(s) = exp\_{p}(\bar{\alpha})(s) and \gamma(s) = exp\_{p}(\bar{\gamma}(s)). We have

l(\alpha) \ge l(\gamma) , where l() denotes the arc length of the corresponding curve.

In addition, if \bar{\alpha}(s) is not a critical point of exp\_{p}, s \in [0,l)] , and if the traces of \alpha and \gamma are distinct, then l(\alpha) > l(\gamma).

(Jacobi) (Thm 1) Let \gamma : [0,l] \to S, \gamma(0) = p be a geodesic without conjugate points; that is, exp\_{p}: T\_{p}(S) \to S is regular at the points of the line \bar{\gamma}(s) = s\gamma’(0) of T\_{p}(S), s \in [0,l] . Let h : [0,l] \times (-\epsilon, \epsilon) \to S be a proper variation of \gamma. Then

There exists a \delta >0 , \delta \le \epsilon, s.t. if t \in (-\delta, \delta), L(t) \ge L(0), where L(t) is the length of the curve h\_{t} : [0,l] \to S that is given by h\_{t}(s) = h(s,t).

If, in addition, the trace of h, is distinct from the trace of \gamma, L(t) > L(0).

(Def 1) Let \gamma : [0,l] \to S be a geodesic of S and let h : [0,l] \times (-\epsilon, \epsilon) \to S be a continuous map with h(s,0) = \gamma(s) , s \in [0,l] . h is said to be a broken variation of \gamma if there exists a partition 0 = s\_{0} < s\_{1} < s\_{2} < \cdots < s\_{n-1} < s\_{n} = l of [0,l] s.t. h: [s\_{i}, s\_{i+1}] \times (-\epsilon, \epsilon) \to S, i = 0,1,…, n-1, is differentiable. The broken variation is said to be proper if h(0,t) = \gamma(0) , h(l,t) = \gamma(l) for every t \in (-\epsilon, \epsilon).

(Lem 2) Let V \in \mathcal{U} be a Jacobi Field along a geodesic \gamma : [0,l] \to S and W \to \mathcal{U}. Then I(V,W) = \langle \frac{DV}{ds}(l) , W(l) \rangle - \langle \frac{DV}{ds} (0), W(0) \rangle.

(Jacobi)(Thm 2) If we let \gamma : [0,l] \to S be a geodesic of S and we let \gamma(s\_{0}) \in \gamma((0,l)) be a point conjugate to \gamma(0) = p relative to \gamma, then there exists a proper broken variation h : [0,l] \times (-\epsilon, \epsilon) \to S of \gamma and a real number \delta >0, \delta \le \epsilon, s.t. if t \in (-\delta, \delta) we have L(t) \le L(0).

(subsection 5.10.) Abstract Surfaces; Further Generalizations

(Def 1) An abstract surface(differentiable manifold of dimension 2) is a set S together with a family of one-to-one maps x\_{\alpha} : U\_{\alpha \to S of open sets U\_{\alpha} \subset R^{2} into S s.t.

\bigcup\_{\alpha}(x\_{\alpha})(U\_{\alpha}) = S.

For each pair \alpha , \beta with x\_{\alpha} (U\_{\alpha}) \cap x\_{\beta}(U\_{\beta}) = W \neq \empty we have that x^{-1}\_{\alpha}(W) , x\_{\beta}^{-1} (W) are open sets in R^{2} , and x\_{\beta}^{-1} \bullet x\_{\alpha}, x\_{\alpha}^{-1} \bullet x\_{\beta} are differentiable maps.

The pair (U\_{\alpha}, x\_{\alpha}) with p \in x\_{\alpha}(U\_{\alpha}) is called a parametrization (or coordinate system) of S around p. x\_{\alpha}(U\_{\alpha}) is called a coordinate neighborhood, and if q = x\_{\alpha} (u\_{\alpha}, v\_{\alpha}) \in S, we say that (u\_{\alpha}, v\_{\alpha}) are coordinates of q in this coordinate system. The family {U\_{\alpha} , x\_{\alpha}} is called a differentiable structure for S.

(Def 2) Let S\_{1} and S\_{2} be abstract surfaces. A map \phi : S\_{1} \to S\_{2} is differentiable at p \in S\_{1} if given a parametrization y : V \subset R^{2} \to S\_{2} around \phi(p) there exists a parametrization x : U \subset R^{2} \to S\_{1} around p s.t. \phi(x(U)) \subset y(V) and the map y^{-1} \bullet \phi \bullet x : x^{-1}(U) \subset R^{2} \to R^{2} is differentiable at x^{-1}(p) . \phi is differentiable on S\_{1} if it is differentiable at every p \in S\_{1} .

(Def 3) A differentiable map \alpha : (-\epsilon, \epsilon) \to S is called a curve on S. Assume that \alpha(0) = p and let D be the set of functions on S which are differentiable at p. The tangent vector to the curve \alpha at t = 0 is the function \alpha’(0) : D \to R given by \alpha’(0) (f) = \frac{d(f \bullet \alpha)}{dt}|\_{t = 0} , f \in D. A tangent vector at a point p \in S is the tangent vector at t = 0 of some curve \alpha : (-\epsilon , \epsilon) \to S with \alpha(0) = p.

(Def 4) Let S\_{1} and S\_{2} be abstract surfaces and let \phi : S\_{1} \to S\_{2} be a differentiable map. For each p \in S\_{1} and each w \in T\_{p}(S\_{1}) , consider a differentiable curve \alpha : (-\epsilon, \epsilon) \to S\_{1} , with \alpha(0) = p, \alpha’(0) = w. Set \beta = \phi \bullet \alpha. The map d\phi\_{p} : T\_{p} (S\_{1}) \to T\_{p}(S\_{2}) given by d\phi\_{p}(w) = \beta’(0) is a well-defined linear map, called the differential of \phi at p.

(Def 5) A geometric surface (Riemannian manifold of dimension 2) is an abstract surface S together with the choice of an inner product \langle , \rangle\_{p} at each T\_{p} (S), p \in S, which varies differentiably with p in the following sense. For some(and hence all) parametrization x : U \to S around p, the funcitons E(u,v) = \langle \frac{ \partial }{\partial u } , \frac{\partial }{\partial u } \rangle , F(u,v) = \langle \frac{\partial }{\partial u} , \frac{\partial }{\partial v} \rangle , G(u,v) = \langle \frac{\partial }{\partial v} , \frac{\partial }{\partial v} \rangle are differentiable functions in U. The inner product \langle , \rangle is often called a (Riemannian) metric on S.

(Def 6) A differentiable map \phi : S \to R^{3} of an abstract surface S into R^{3} is an immersion if the differential d \phi\_{p} : T\_{p} (S) \to T\_{p}(R^{3}) is injective. If, in addition, S has a metric \langle, \rangle and \langle d \phi\_{p}(v), d \phi\_{p}(w) \rangle\_{\phi(p)} = \langle v , w \rangle\_{p} , v , w \in T\_{p}(S), \phi is said to be an isometric immersion.

(Def 7) Let S be an abstract surface. A differentiable map \phi : S \to R^{n} is an embedding if \phi is an immersion and a homeomorphism onto its image.

(Def 1a) A differentiable manifold of dimension n is a set M together with a family of one-to-one maps x\_{\alpha} : U\_{\alpha} \to M of open sets U\_{\alpha} \subset R^{n} into M s.t.

\bigcup\_{\alpha}(x\_{\alpha})(U\_{\alpha}) = M.

For each pair \alpha , \beta with x\_{\alpha} (U\_{\alpha}) \cap x\_{\beta}(U\_{\beta}) = W \neq \empty we have that x^{-1}\_{\alpha}(W) , x\_{\beta}^{-1} (W) are open sets in R^{n} , and x\_{\beta}^{-1} \bullet x\_{\alpha}, x\_{\alpha}^{-1} \bullet x\_{\beta} are differentiable maps.

The family {U\_{\alpha}, x\_{\alpha}} is maximal relative to first and second conditions.

The family {U\_{\alpha}, x\_{\alpha}} satisfying first and second conditions is called a differentiable structure on M.

(Def 5a) ) A Riemannian manifold is an n-dimensional differentiable manifold M together with a choice, for each p \in M, of an inner product \langle , \rangle\_{p} at each T\_{p} (M), p \in M, which varies differentiably with p in the following sense. For some parametrization x\_{\alpha} : U\_{\alpha} \to M with p \in x\_{\alpha}(U\_{\alpha}) , the functions g\_{ij}(u\_{1},…,u\_{n}) = \langle \frac{\partial}{\partial u\_{i}}, \frac{\partial}{\partial u\_{j}} \rangle , i, j = 1, …, n , are differentiable at x\_{\alpha}^{-1} (p) ; here (u\_{1}, …, u\_{n}) are the coordinates of U\_{\alpha} \subset R^{n}.

(subsection 5.11) Hilbert’s Theorem

(Thm) A complete geometric surface S with constant negative curvature cannot be isometrically immersed in R^{3}.

(Def) assume the curvature K \equiv -1. Since exp\_{p} : T\_{p}(S) \to S is a local diffeomorphism , it induces an inner product in T\_{p}(S). Denote by S’ the geometric surface T\_{p}(S) with this inner product.

(Lem 1) The area of S’ is infinite.

(Def) Assume there exists an isometric immersion \phi : S’ \to R^{3} , where S’ is a geometric surface homeomorphic to a plane and with K \equiv -1.

(Lem 2) For each p \in S’ there is a parametrization x : U \subset R^{2} \to S’, p \in x(U) , s.t. the coordinate curves of x are the asymptotic curves of x(U) = V’ and form a Tchbyshef net. (we shall express this by saying that the asymptotic curves of V’ form a Tchebyshef net.)

(Lem 3) Let V’ \subset S’ be a coordinate neighborhood of S’ s.t. the coordinate curves are the asymptotic curves in V’. Then the area A of any quadrilateral formed by the coordinate curves is smaller than 2\pi.

(Lem 4) For a fixed t, the curve x(s,t) , -\infty < s < \infty , is an asymptotic curve with s as arc length.

(Lem 5) x is a local diffeomorphism.

(Lem 6) x is surjective.

(Lem 7) On S’ there are two differentiable linearly independent vector fields which are tangent to the asymptotic curves of S’.

(Lem 8) x is injective.

(Appendix) Point-Set Topology of Euclidean Spaces

(subsection A) Preliminaries

(Def 1) A sequence p\_{1}, …, p\_{i}, …, \in R^{n} converges to p\_{0} \in R^{n} if given \epsilon >0, there exists an index i\_{0} of the sequence s.t. p\_{i} \in B\_{e}(p\_{0}) for all i > i\_{0} . In this situation, p\_{0} is the limit of the sequence {p\_{i}} and this is denoted by {p\_{i}} \to p\_{0}.

(Prop 1) A map F : U \subset R^{n} \to R^{m} is continuous at p\_{0} \in U iff for each converging sequence {p\_{i}} \to p\_{0} in U, the sequence {F(p\_{i})} converges to F(p\_{0}).

(Def 2) A point p \in R^{n} is a limit point of a set A \subset R^{n} if every neighborhood of p in R^{n} contains one point of A distinct from p.

(Def 3) A set F \subset R^{n} is closed if every limit point of F belongs to F. The closure of A \subset R^{n} denoted by \bar{A} is the union of A with its limit points.

(Prop 2) F \subset R^{n} is closed iff the complement R^{n} – F of F is open.

(Prop 3) A map F : U \subset R^{n} \to R^{m} is continuous iff for each open set V \subset R^{m} , F^{-1}(V) is an open set.

(Cor) F : U \subset R^{n} \to R^{m} is continuous iff for every closed set A \subset R^{m}, F^{-1} (A) is a closed set.

(Def 4) Let A \subset R^{n}. The boundary Bd A of A is the set of points p in R^{n} s.t. every neighborhood of p contains points in A and points in R^{n} – A.

(Def 5) Let A \subset R^{n} . We say that V \subset A is an open set in A if there exists an open set U in R^{n} s.t. V = U \cap A. A neighborhood of p \in A in A is an open set in A containing p.

(Def 6) A subset A \subset R of the real line R is bounded above if there exists M \in R s.t. M \ge a for all a \in A. The number M is called an upper bound for A. When A is bounded above, a supremum or a least upper bound of A, sup A(or l.u.b. A) is an upper bound M which satisfies the following condition : Given \epsilon >0, there exists a \in A s.t. M - \epsilon < a. By changing the sign of the above inequalities, we define similarly a lower bound for A and an infimum (or a greatest lower bound) of A, inf A (or g.l.b. A).

(Axiom of Completeness of Real numbers) Let A \subset R be nonempty and bounded above(below). Then there exists sup A (inf A).

(Lem 1) Call a sequence {x\_{i}} of real numbers a Cauchy sequence if given \epsilon <0, there exists i\_{0} s.t. |x\_{i} – x\_{j}| < \epsilon for all i,j > i\_{0}. A sequence is convergent iff it is a Cauchy sequence.

(Def 7) A sequence {p\_{i}} , p\_{i} \in R^{n}, is a Cauchy sequence if given \epsilon >0, there exists an index i\_{0} s.t. the distance |p\_{i} – p\_{j}| < \epsilon for all i,j > i\_{0}.

(Prop 4) A sequence {p\_{i}} , p\_{i} \in R^{n}, converges iff it is a Cauchy sequence.

(subsection B) Connected sets

(Def 8) A continuous curve \alpha : [a,b] \to A \subset R^{n} is called an arc in A joining \alpha(a) to \alpha(b).

(Def 9) A \subset R^{n} is arcwise connected if , given two points p, q \in A, there exists an arc in A joining p to q.

(Def 10) A \subset R^{n} is connected when it is not possible to write A = U\_{1} \cap U\_{2} , where U\_{1} and U\_{2} are nonempty open sets in A and U\_{1} \cap U\_{2} = \empty.

(Prop 5) Let A \subset R^{n} be connected and let B \subset A be simultaneously open and closed in A. Then either B = \empty or B = A.

(Prop 6) Let F : A \subset R^{n} \to R^{m} be continuous and A be connected. Then F(A) is connected.

(Def 11) An interval of the real line R is any of the sets a < x < b, a \le x \le b, a < x \le b , a \le x < b , x \in R, The cases a = b , a = - \infty , b = + \infty are not excluded, so that an interval may be a point , a half-line, or R itself.

(Prop 7) A \subset R is connected iff A is an interval.

(Prop 8) Let f : A \subset R^{n} \to R be continuous and A be connected. Assume that f(q) \neq 0 for all q \in A. Then f does not change sign in A.

(Prop 9) Let A \subset R^{n} be arcwise connected. Then A is connected.

(Def 12) A set A \subset R^{n} is locally arcwise connected if for each p \in A and each neighborhood V of p in A there exists an arcwise connected neighborhood U \subset V of p in A.

(Prop 10) Let A \subset R^{n} be a locally arcwise connected set. Then A is connected iff it is arcwise connected.

(subsection C) Compact sets

(Def 13) A set A \subset R^{n} is bounded if it is contained in some ball of R^{n}. A set K \subset R^{n} is compact if it is closed and bounded.

(Def 14) An open cover of a set A \subset R^{n} is a family of open sets {U\_{\alpha}} , \alpha \in \mathfrak{A} s.t. \bigcup\_{\alpha} U\_{\alpha} = A. When there are only finitely many U\_{\alpha} in the family, we say that the cover is finite. If the subfamily {U\_{\beta}} , \beta \in \mathfrak{B} \subset \mathfrak{A} , still covers A, that is, \bigcup\_{\beta} U\_{\beta{ = A, we say that {U\_{\beta}} is a subcover of {U\_{\alpha}} .

(Prop 11) For a set K \subset R^{n} the following assertions are equivalent.

K is compact.

(Heine-Borel) Every open cover of K has a finite subcover.

(Bolzano-Weierstrass) Every infinite subset of K has a limit point in K.

(Prop 12) Let F : K \subset R^{n} \to R^{m} be continuous and let K be compact. Then F(K) is compact.

(Prop 13) Let f : K \subset R^{n} \to R be a continuous function defined on a compact set K. Then there exists p\_{1}, p\_{2} \in K s.t. f(p\_{2}) \le f(p) \le f(p\_{1}) for all p \in K; that is, f reaches a maximum at p\_{1} and a minimum at p\_{2}.

(subsection D) Connected Components

(Prop 14) Let C\_{\alpha} \subset R^{n} be a family of connected sets s.t. \bigcap\_{\alpha} C\_{\alpha} \neq \empty. Then \bigcup\_{\alpha} C\_{\alpha} = C is a connected set.

(Def 15) Let A \subset R^{n} and p \in A. The union of all connected subsets of A which contain p is called the connected component of A containing p.

(Prop 15) Let C \subset A \subset R^{n} be a connected set. Then the closure \bar{C} of C in A is connected.

(Cor) A connected component C \subset A \subset R^{n} of a set A is closed in A.

(prop 16) Let C \subset A \subset R^{n} be a connected component of a locally arcwise connected set A. Then C is open in A.